(Yummy, yummy, yummy, I've got)

Theoretical Atom Optics

(in my tummy)

Lectures for the ACQAO Summer School

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"There has been an alarming increase in the number of things I do not know"

Review

Complete description of multiple bosonic atoms

– The basis states of the total Hilbert space have a given occupation number for each single particle mode:

$$|\psi\rangle = \sum_{n_1,n_2,n_3,\cdots} f_{n_1,n_2,n_3,\cdots} |n_1,n_2,n_3,\cdots\rangle$$

Typical Hamiltonian:

$$\hat{H} = \int d\mathbf{x}\, \hat{\psi}^{\dagger}(\mathbf{x})\left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x})\right)\hat{\psi}(\mathbf{x}) + U\int d\mathbf{x}\, \hat{\psi}^{\dagger}(\mathbf{x})\hat{\psi}^{\dagger}(\mathbf{x})\hat{\psi}(\mathbf{x})\hat{\psi}(\mathbf{x})$$

Major classes of approximation

Quantum field theory calculations are hard

Approximations ignore either:

- Complexity in the quantum state of each mode
- The number of modes
- Systems with strong interactions

Perturbative methods: incredibly refined in the world of QED, some branches of condensed matter physics and particle physics. Unfortunately, atom optics is typically highly non-perturbative

Single mode approximation

Ignore all the irrelevant modes

- Very common in quantum optics
- Applicable to spatially "simple" systems
- Depends on initial state and Hamiltonian

$$\hat{H} = \int d\mathbf{x}\, \hat{\psi}^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x}) \right) \hat{\psi}(\mathbf{x}) + U \int d\mathbf{x}\, \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}(\mathbf{x}) \hat{\psi}(\mathbf{x})$$

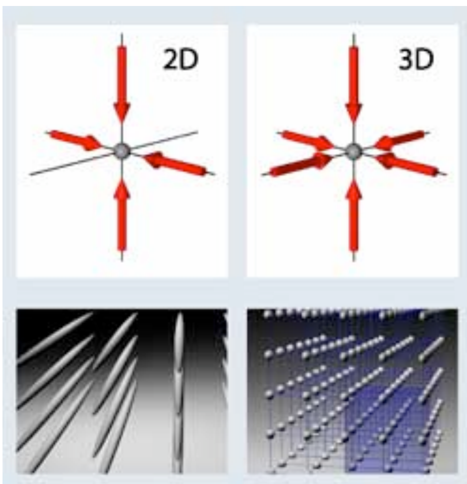
$$\hat{H} = \sum_{jk} T_{jk}\, \hat{b}_j^\dagger \hat{b}_k + \sum_{jklm} V_{jklm}\, \hat{b}_j^\dagger \hat{b}_k^\dagger \hat{b}_l \hat{b}_m$$

In some well-chosen basis, we may be able to consider only a finite (small) number of these modes

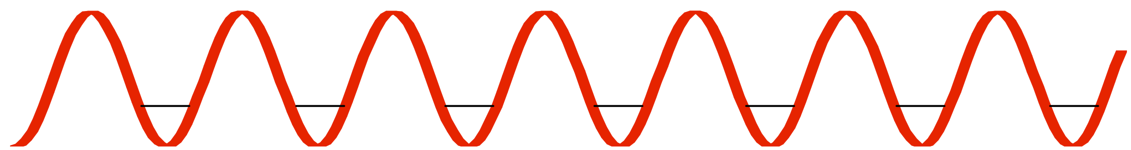
e.g. $$\hat{H} = E_1\, \hat{b}_1^\dagger \hat{b}_1 + \kappa \left(\hat{b}_1^\dagger \hat{b}_2 + \hat{b}_2^\dagger \hat{b}_1 \right) + V \left(\hat{b}_1^\dagger \hat{b}_1^\dagger \hat{b}_1 \hat{b}_1 + \hat{b}_2^\dagger \hat{b}_2^\dagger \hat{b}_2 \hat{b}_2 \right)$$

Optical lattices

Optical lattices make periodic potentials for atoms



http://www.physik.uni-mainz.de/quantum/bec/experiments/opticallattices.html



$$\hat{a}_{j-2} \quad \hat{a}_{j-1} \quad \hat{a}_{j} \quad \hat{a}_{j+1} \quad \hat{a}_{j+2} \quad \hat{a}_{j+3}$$

$$\hat{H} = \int d\mathbf{x}\, \hat{\psi}^{\dagger}(\mathbf{x}) \left(-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x}) \right) \hat{\psi}(\mathbf{x})$$

$$+U \int d\mathbf{x}\, \hat{\psi}^{\dagger}(\mathbf{x}) \hat{\psi}^{\dagger}(\mathbf{x}) \hat{\psi}(\mathbf{x}) \hat{\psi}(\mathbf{x})$$

$$H \approx -J \sum_{j} \hat{a}_{j}^{+} \hat{a}_{j+1} + \frac{1}{2} U \sum_{i} a_{i}^{+} a_{i} \left(a_{i}^{+} a_{i} - 1 \right)$$

Bose-Hubbard model

Atom laser

We saw yesterday a semiclassical pumped atom laser model reaching a nice, stable steady state.

Let's build a *quantum* model of this system:

- Two electronic components
- Different potential for each component
- Interactions between atoms
- Coupling between the two components
- Pumping into the trapped component

Atom laser

$$\hat{H} = \int d\mathbf{x}\, \hat{\psi}_t^\dagger(\mathbf{x})\left(-\frac{\hbar^2}{2m}\nabla^2 + V_t(\mathbf{x})\right)\hat{\psi}_t(\mathbf{x}) + U_{tt}\int d\mathbf{x}\, \hat{\psi}_t^\dagger(\mathbf{x})\hat{\psi}_t^\dagger(\mathbf{x})\hat{\psi}_t(\mathbf{x})\hat{\psi}_t(\mathbf{x})$$

$$+\int d\mathbf{x}\, \hat{\psi}_u^\dagger(\mathbf{x})\left(-\frac{\hbar^2}{2m}\nabla^2 + V_u(\mathbf{x})\right)\hat{\psi}_u(\mathbf{x}) + U_{uu}\int d\mathbf{x}\, \hat{\psi}_u^\dagger(\mathbf{x})\hat{\psi}_u^\dagger(\mathbf{x})\hat{\psi}_u(\mathbf{x})\hat{\psi}_u(\mathbf{x})$$

$$+U_{tu}\int d\mathbf{x}\, \hat{\psi}_t^\dagger(\mathbf{x})\hat{\psi}_u^\dagger(\mathbf{x})\hat{\psi}_u(\mathbf{x})\hat{\psi}_t(\mathbf{x})$$

$$+\int d\mathbf{x}\, \kappa(\mathbf{x})e^{-i\mathbf{k}.\mathbf{x}}\,\hat{\psi}_t^\dagger(\mathbf{x})\,\hat{\psi}_u(\mathbf{x}) + \kappa^*(\mathbf{x})e^{i\mathbf{k}.\mathbf{x}}\,\hat{\psi}_u^\dagger(\mathbf{x})\,\hat{\psi}_t(\mathbf{x})$$

$$+\hat{H}_{\text{pump}}(\hat{\psi}_t,\hat{\psi}_t^\dagger)$$

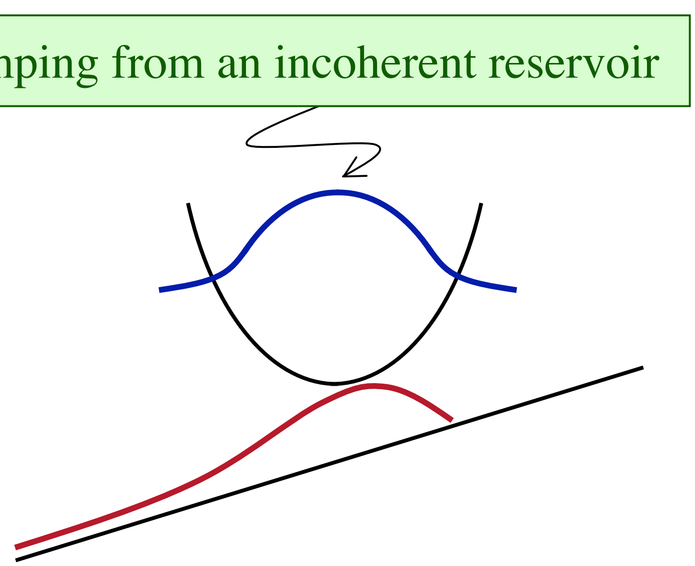
Excellent model

Way too hard to solve

The trouble with semi-classical

Yesterday, we made the semiclassical approximation
- Are the quantum features important?
- What is the most interesting feature of an atom beam?

Suppose a semi-classical model reaches steady state:



Almost by definition, the far field is in an eigenstate.

Semiclassical model predicts infinitely narrow linewidth

Change trapped basis

$$\hat{H} = \int d\mathbf{x}\, \hat{\psi}_t^\dagger(\mathbf{x})\left(-\frac{\hbar^2}{2m}\nabla^2 + V_t(\mathbf{x})\right)\hat{\psi}_t(\mathbf{x}) + U_{tt}\int d\mathbf{x}\, \hat{\psi}_t^\dagger(\mathbf{x})\hat{\psi}_t^\dagger(\mathbf{x})\hat{\psi}_t(\mathbf{x})\hat{\psi}_t(\mathbf{x})$$

$$+\int d\mathbf{x}\, \hat{\psi}_u^\dagger(\mathbf{x})\left(-\frac{\hbar^2}{2m}\nabla^2 + V_u(\mathbf{x})\right)\hat{\psi}_u(\mathbf{x}) + U_{uu}\int d\mathbf{x}\, \hat{\psi}_u^\dagger(\mathbf{x})\hat{\psi}_u^\dagger(\mathbf{x})\hat{\psi}_u(\mathbf{x})\hat{\psi}_u(\mathbf{x})$$

$$+U_{tu}\int d\mathbf{x}\, \hat{\psi}_t^\dagger(\mathbf{x})\hat{\psi}_u^\dagger(\mathbf{x})\hat{\psi}_u(\mathbf{x})\hat{\psi}_t(\mathbf{x})$$

$$+\int d\mathbf{x}\left(\kappa(\mathbf{x})e^{-i\mathbf{k}.\mathbf{x}}\,\hat{\psi}_t^\dagger(\mathbf{x})\,\hat{\psi}_u(\mathbf{x}) + \kappa^*(\mathbf{x})e^{i\mathbf{k}.\mathbf{x}}\,\hat{\psi}_u^\dagger(\mathbf{x})\,\hat{\psi}_t(\mathbf{x})\right)$$

$$+\hat{H}_{\text{pump}}(\hat{\psi}_t,\hat{\psi}_t^\dagger)$$

Choose basis for trapped atoms

$$\hat{\psi}_t(x) = \sum_j u_j(x)\hat{b}_j \qquad \hat{b}_j = \int d\mathbf{x}\, u_j^*(\mathbf{x})\hat{\psi}_t(\mathbf{x})$$

New atom laser Hamiltonian

$$\hat{H} = \sum_{jk} T_{jk}\, \hat{b}_j^\dagger \hat{b}_k + \sum_{jklm} V_{jklm}\, \hat{b}_j^\dagger \hat{b}_k^\dagger \hat{b}_l \hat{b}_m$$

$$+ \int d\mathbf{x}\, \hat{\psi}_u^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m} \nabla^2 + V_u(\mathbf{x}) \right) \hat{\psi}_u(\mathbf{x}) + U_{uu} \int d\mathbf{x}\, \hat{\psi}_u^\dagger(\mathbf{x}) \hat{\psi}_u^\dagger(\mathbf{x}) \hat{\psi}_u(\mathbf{x}) \hat{\psi}_u(\mathbf{x})$$

$$+ U_{tu} \sum_{jk} \int d\mathbf{x}\, u_j^*(\mathbf{x})\, u_k(\mathbf{x})\, \hat{b}_j^\dagger \hat{\psi}_u^\dagger(\mathbf{x}) \hat{\psi}_u(\mathbf{x})\, \hat{b}_k$$

$$+ \sum_j \int d\mathbf{x} \left(\kappa(\mathbf{x}) e^{-i\mathbf{k.x}}\, u_j^*(\mathbf{x}) \hat{b}_j^\dagger\, \hat{\psi}_u(\mathbf{x}) + \kappa^*(\mathbf{x}) e^{i\mathbf{k.x}}\, u_j(\mathbf{x}) \hat{\psi}_u^\dagger(\mathbf{x}) \hat{b}_k \right)$$

$$+ \hat{H}_{\mathrm{pump}}(\hat{b}_k, \hat{b}_j^\dagger)$$

$$T_{jk} = \int d\mathbf{x}\, u_j^*(\mathbf{x}) \left(-\frac{\hbar^2}{2m} \nabla^2 + V_t(\mathbf{x}) \right) u_k(\mathbf{x}) \qquad V_{jklm} = U_{tt} \int d\mathbf{x}\, u_j^*(\mathbf{x}) u_k^*(\mathbf{x})\, u_l(\mathbf{x}) u_m(\mathbf{x})$$

Did we choose well?

The hope is that most of the $\hat{b}_k$ modes can be ignored.

$$\hat{H} = \hbar\omega_0 \hat{b}_0^\dagger \hat{b}_0 + U_0 \hat{b}_0^\dagger \hat{b}_0^\dagger \hat{b}_0 \hat{b}_0$$

$$+\int d\mathbf{x}\, \hat{\psi}_u^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m}\nabla^2 + V_u(\mathbf{x}) \right) \hat{\psi}_u(\mathbf{x}) + U_{uu} \int d\mathbf{x}\, \hat{\psi}_u^\dagger(\mathbf{x}) \hat{\psi}_u^\dagger(\mathbf{x}) \hat{\psi}_u(\mathbf{x}) \hat{\psi}_u(\mathbf{x})$$

$$+U_{tu} \int d\mathbf{x}\, |u_0(\mathbf{x})|^2 \hat{b}_0^\dagger \hat{b}_0 \hat{\psi}_u^\dagger(\mathbf{x}) \hat{\psi}_u(\mathbf{x})$$

$$+\int d\mathbf{x} \left(\kappa(\mathbf{x}) e^{-i\mathbf{k}.\mathbf{x}}\, u_0^*(\mathbf{x}) \hat{b}_0^\dagger\, \hat{\psi}_u(\mathbf{x}) + \kappa^*(\mathbf{x}) e^{i\mathbf{k}.\mathbf{x}}\, u_0(\mathbf{x}) \hat{\psi}_u^\dagger(\mathbf{x}) \hat{b}_0 \right)$$

$$+\hat{H}_{\text{pump}}(\hat{b}_0, \hat{b}_0^\dagger)$$

We have reduced the number of indices. Is it soluble?

The single mode atom laser

The dynamics of the output quantum field $\hat{\psi}_u(\mathbf{x})$ are still complicated. How do we simplify the dynamics?

- Ignore interactions in the output field
- Ignore interactions with output field
- Choose eigenstate basis for output field

$$\hat{\psi}_u(x) = \int d\mathbf{p}\, w(\mathbf{x},\mathbf{p})\hat{c}(\mathbf{p}) \qquad \hat{c}(\mathbf{p}) = \int d\mathbf{x}\, w^*(\mathbf{x},\mathbf{p})\hat{\psi}_u(\mathbf{x})$$

$$\hat{H} = \hbar\omega_0 \hat{b}_0^\dagger \hat{b}_0 + U_0 \hat{b}_0^\dagger \hat{b}_0^\dagger \hat{b}_0 \hat{b}_0 + \int d\mathbf{p}\, \hbar\omega_{\mathbf{p}}\, \hat{c}_{\mathbf{p}}^\dagger \hat{c}_{\mathbf{p}}$$
$$+ \hbar \int d\mathbf{p} \left(\chi(\mathbf{p}) \hat{b}_0^\dagger\, \hat{c}_{\mathbf{p}} + \chi^*(\mathbf{p}) \hat{b}_0\, \hat{c}_{\mathbf{p}}^\dagger \right) + \hat{H}_{\text{pump}}(\hat{b}_0, \hat{b}_0^\dagger)$$

The single mode atom laser

We have simplified the output dynamics, but we still can't model them numerically.

Is there an analytical result?

- Output modes all come from a simple interaction term

$$i\frac{\partial \hat{c}_{\mathbf{p}}}{\partial t} = \omega_{\mathbf{p}}\,\hat{c}_{\mathbf{p}} + \chi^{*}(\mathbf{p})\,\hat{b}_0$$

$$\hat{c}_{\mathbf{p}}(t) = \hat{c}_{\mathbf{p}}(0)\,e^{-i\omega_{\mathbf{p}}\,t} - i\,\chi^{*}(\mathbf{p})\int_0^t e^{-i\omega_{\mathbf{p}}\,(t-u)}\,\hat{b}_0(u)$$

Atom laser output

The spectrum of the output atom laser flux

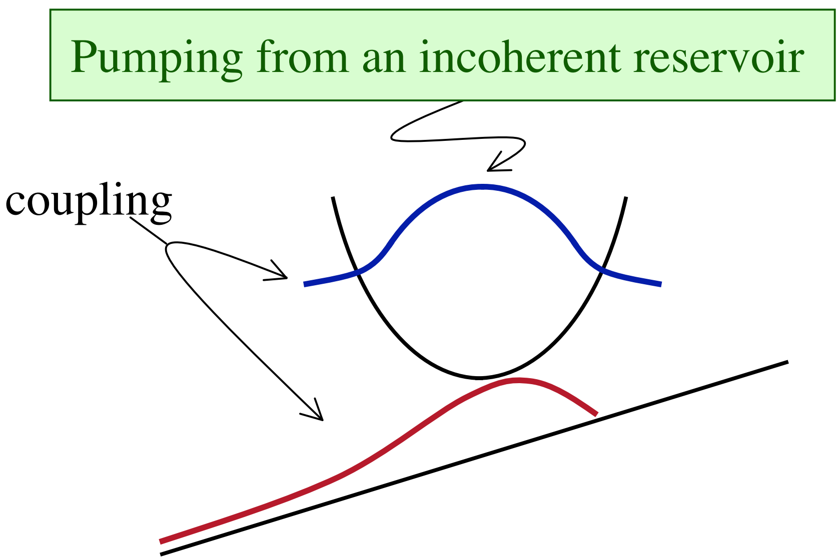
$$\frac{\partial\left\langle \hat{c}_{\mathbf{p}}^{\dagger}\hat{c}_{\mathbf{p}}\right\rangle}{\partial t} = \left\langle \hat{c}_{\mathbf{p}}^{\dagger}(0)\hat{c}_{\mathbf{p}}(0)\right\rangle + \left\langle \hat{B}_{lah}\,\hat{c}_{\mathbf{p}}(0)\right\rangle + \left\langle \hat{c}_{\mathbf{p}}^{\dagger}(0)\,\hat{J}_{unk}\right\rangle$$

$$+2\left|\chi(\mathbf{p})\right|^{2}\Re e\left(\int_{0}^{t} du\, e^{-i\omega_{\mathbf{p}}(t-u)}\left\langle \hat{b}_{0}^{\dagger}(t)\hat{b}_{0}(u)\right\rangle\right)$$

All interesting observables depend only on the dynamics of the lasing mode!

The atom laser

Looking back at our simple model:



Assuming a single, self-consistent mode in the trap, the lasing mode, and low output density, we can solve the problem without *any* further approximations

The dynamics of the lasing mode depend on the trap Hamiltonian, back action from the output and the pumping process.

Exercise

$$\frac{\partial\left\langle \hat{c}_{\mathbf{p}}^{\dagger}\hat{c}_{\mathbf{p}}\right\rangle}{\partial t} = \left\langle \hat{c}_{\mathbf{p}}^{\dagger}(0)\hat{c}_{\mathbf{p}}(0)\right\rangle + \left\langle \hat{B}_{lah}\,\hat{c}_{\mathbf{p}}(0)\right\rangle + \left\langle \hat{c}_{\mathbf{p}}^{\dagger}(0)\,\hat{J}_{unk}\right\rangle$$

$$+2\left|\chi(\mathbf{p})\right|^{2}\Re e\left(\int_{0}^{t} du\, e^{-i\omega_{\mathbf{p}}(t-u)}\left\langle \hat{b}_{0}^{\dagger}(t)\hat{b}_{0}(u)\right\rangle\right)$$

Assume the lasing mode is in a coherent state

$$\left|\psi\right\rangle_{\text{lasing mode}} = \left|\beta\, e^{-i\omega_0 t}\right\rangle$$

Q. What is the linewidth of the atom laser if there are initially no atoms in the output?

The pumped (atom) laser

- The linewidth of a laser is limited by "quantum noise"
- The model we developed works equally well for an optical laser
- The remaining step is to calculate the dynamics of the lasing mode

Optical outcoupling is blessed by the constant (high) speed of light. This leads to Markovian models, where there is no back action.

$$\left.\frac{\partial \hat{b}_0}{\partial t}\right|_{\substack{\text{Markovian}\\ \text{damping}}} = -\frac{\gamma}{2}\hat{b}_0$$

Atomic lasers deal with atoms taking significant time to get out of the coupling region

$\Rightarrow$ This leads to non-Markovian models

$$\left.\frac{\partial \hat{b}_0}{\partial t}\right|_{\text{outcoupling}} = \int_0^t du\, f(t-u)\hat{b}_0(u)$$

The pumped (atom) laser

- The properties of the beam depend very strongly on the details of the pumping process. They are all theoretical for atom lasers
- The goal is to produce stable phase in the lasing mode

$$\frac{\partial\left\langle \hat{c}_{\mathbf{p}}^{\dagger}\hat{c}_{\mathbf{p}}\right\rangle}{\partial t} \propto \Re e\left(\int_0^t du\, e^{-i\omega_{\mathbf{p}}(t-u)}\left\langle \hat{b}_0^{\dagger}(t)\hat{b}_0(u)\right\rangle\right)$$

$$g(\tau) \equiv \lim_{t\to\infty}\left\langle \hat{b}_0^{\dagger}(t+\tau)\hat{b}_0(t)\right\rangle$$

a.k.a. two-time correlation

temporal coherence

For a highly pumped system with a wise choice of pumping mechanism:

$$g(\tau) \approx e^{i(\omega_0+\Delta)\tau} e^{-r|\tau| / 4(\bar{n}+n_s)^2}$$

r	pumping rate
$\bar{n}$	mean lasing mode number
n_s	spontaneous loss

The gain-narrowed (atom) laser

Output flux $\propto r \propto \bar{n}$ Output linewidth $\propto \dfrac{r}{4\left(\bar{n}+n_s\right)^2} \propto \dfrac{K}{\bar{n}}$

With the right pumping mechanism (must be continuous), the output spectrum narrows as the flux increases!

This is called gain-narrowing, and we want it

Requirements

- Continuous pumping mechanism (irreversible)
- Stable, single mode operation

Summary

Quantum field theory fundamentals - how to build models

Semiclassical approximation - how to build models

- www.xmds.org(the numerics)

Single mode approximations - how to build models

Atom laser models as an example of each

- current experimental models
- pumped atom laser multimode stability
- feedback (needs detection)
- gain-narrowing (needs continuous pumping)