We need a microscopic model that includes:

- Feshbach resonance in the two-body potential
- BCS Cooper pair formation
- BCS-BEC crossover as interaction increases

The model (due to Timmermans, Holland, Drummond and coworkers) explicitly includes the Fermi atoms, the Bosonic molecules formed from these atoms, and the Feshbach resonance coupling term. This theory is often now called resonance superfluidity, a term introduced by Holland.
Feshbach resonance: two body physics

Stable molecules form when \( a_{2b} > 0 \). This is equivalent to \( 2\nu < 0 \) or \( B < B_0 \).

\[
2\nu \propto B - B_0
\]

\[
\frac{4\pi\hbar^2 a_{2b}}{m} \equiv U + \frac{g^2}{2\nu} \quad \text{---} \quad a_s = a_{bg} \left(1 + \frac{w}{B_0 - B}\right)
\]
The atom-molecule interaction is denoted by $g_r$

The non-resonant attractive interaction is $-U$

The molecular bound state energy $2\nu$ can be tuned. Molecules (with finite lifetime) start to form when $2\nu \leq 2\varepsilon_F$ and will not be able to decay when $2\nu < 0$. 

$$\mathcal{H} = \sum_{p,\sigma} \varepsilon_p c_{p,\sigma}^\dagger c_{p,\sigma} + \sum_{q} (E_q^0 + 2\nu) b_{q,\sigma}^\dagger b_{q,\sigma}$$

$$- U \sum_{p,\sigma} c_{p,\uparrow}^\dagger c_{p,\downarrow}^\dagger c_{p',\downarrow}^\dagger c_{p',\uparrow} + g_r \sum_{p,q} [b_{q}^\dagger c_{p+q/2,\uparrow} c_{p+q/2,\downarrow} + h.c].$$

$$E_q^0 = \frac{q^2}{2M}$$

\text{FERMI ATOMS} \quad \text{MOLECULES (b-BOSONS)}

\text{INTERACTION VERTEX $g$}
A crucial feature of this Hamiltonian is that the b-molecules are formed from the Fermi atoms. There is thus only one chemical potential, with

\[ H - \mu N = H - \mu N_F - 2\mu N_B \text{, with } \mu_B = 2\mu. \]

This coupled Hamiltonian modifies the effect of the bare two-body Feshbach resonance. Two atoms are now part of an interacting system in the presence of a filled Fermi sea.
First thing to do is to solve our coupled FB model in a mean field approximation, allowing for Cooper pairs and a Bose condensate \((q = 0)\) of b-molecules:

\[
H_{FB} - \mu N = \sum_{p,\sigma} (\varepsilon_{p,\sigma} - \mu) c_{p,\sigma}^+ c_{p,\sigma} + \phi_m^2 (E_{q=0} - 2\nu - 2\mu) \\
- U \sum_p (\phi_C c_{p\uparrow}^+ c_{-p\downarrow}^+ + h.c.) + g \sum_p (\phi_m c_{p\uparrow}^+ c_{-p\downarrow}^+ + h.c.)
\]

\(\phi_C = \) Cooper pair condensate - see previous lectures  
\(\phi_m = \) Molecular condensate = \(\langle b_{q=0} \rangle\)

Both condensates are dependent on each other. We end up with a BCS-type theory but now with a composite order parameter:

\[
\tilde{\Delta} = U\phi_C - g\phi_m
\]
This order parameter is the sum of contributions from two mechanisms:

Pair wavefunction = scattering channel + molecular channel

However, they are strongly coupled to each other and one determines the other:

\[ \phi_m = -\frac{g}{2\nu - 2\mu} \phi_c \]

The number of Bose condensed b-molecules is given by \[ N_b = |\phi_m|^2 . \]
The composite BCS order parameter reduces to:

\[ \tilde{\Lambda} = U \phi_C - g \phi_m = \left( U + \frac{g^2}{2\nu - 2\mu} \right) \phi_C \equiv U_{\text{eff}} \phi_C \]

The physics is clear. The attractive interaction - $U$ between the Fermi atoms in the open channel is now renormalized to $U_{\text{eff}}$ by the resonant coupling to the b-molecules in the closed channel. Calculation also shows that we always have $2\nu > 2\mu$.

One can speak in terms of a Bose condensate of BCS Cooper pairs or in terms of a molecular BEC of b-molecules, on both sides of the Feshbach resonance.
Note we are now dealing with a renormalized Feshbach resonance for atoms interacting in a superfluid Fermi gas, not two atoms in a vacuum. The b-molecules are described by a propagator

\[ D_0(q, \omega) = \frac{1}{\omega - \left( E_q^0 + 2\nu - 2\mu \right)} \]

For coupling to Cooper pairs with \( q = 0, \omega = 0 \), this b-molecule propagator reduces to

\[ D_0(0,0) = -\frac{1}{(2\nu - 2\mu)} \]

The b-molecules play the role of phonons in metals, leading to a large pairing interaction as \( (2\nu - 2\mu) \to 0 \).
It is no surprise that the renormalized energy gap is given by a BCS-type gap equation but now with an enhanced attractive interaction $U_{\text{eff}}$

$$\tilde{\Delta}(T) = \left[ U + \frac{g^2}{2\nu - 2\mu} \right] \sum_k \frac{\tilde{\Delta}(T)}{2E_k} \tanh \left( \frac{1}{2\beta E_k} \right)$$

$$E_k = \sqrt{[\varepsilon_k - \mu]^2 + \tilde{\Delta}^2} = \text{BCS quasiparticle spectrum with energy gap at } \tilde{\Delta}$$

At $T = T_{\text{BCS}}$, $\tilde{\Delta}(T) \rightarrow 0$ and above equation reduces to BCS equation for $T_{\text{BCS}}$,

$$1 = \left[ U + \frac{g^2}{2\nu - 2\mu} \right] \sum_k \frac{\tanh(E_k/2k_BT_{\text{BCS}})}{2(\varepsilon_k - \mu)}$$

We already see the possibility that $T_C$ will be large. However, $\mu$ is also dependent on the value of $2\nu$. 
The relative weight of the b-molecules and the Cooper pairs in the composite Bose condensate is shown as a function of the temperature (uniform gas).

*Ohashi and Griffin*, PRA, 2003
Uniform gas results at $T_c$

One has long-lived $b$-molecules on the BCS side ($\nu > 0$) and stable Cooper pairs on the BEC side ($\nu < 0$).

Similar results are obtained at $T < T_c$ and in trapped gases.
Chemical potential (in units of the trap frequency $\omega_0$) at $T = 0$. 

How parameters change through crossover

**BEC** vs **BCS**

- $a_s > 0$
- $2\nu < 2\varepsilon_F$
- $\mu < 0$

Energy gap in uniform gas $\rightarrow$

$$E_g = [\mu^2 + \Delta^2]^{1/2}$$

Momentum distribution spread out

- $a_s < 0$
- $2\nu > 2\varepsilon_F$
- $\mu > 0 \ (\approx \varepsilon_F)$

Energy gap

$$E_g = \Delta$$

Momentum distribution has sharp Fermi surface
To calculate the chemical potential $\mu$ and the order parameter $\Delta$ in a self-consistent way as a function of $\nu$, one has to include the fluctuations around the BCS - Gorkov MFA:

- The Cooper pairs outside the BCS condensate
  - Nozieres and Schmitt-Rink (1985) at $T_c$.
- The b-molecules outside the molecular condensate
  - Ohashi and Griffin (2002) at $T_c$.
- Both effects included below $T_c$ by O&G (2003).

The number of b-molecules and Cooper pairs is self-consistently adjusted as $2\nu$ is decreased,

$$N_F = N_{\text{atoms}} + 2N_{\text{Cooper pairs}} + 2N_{\text{b-molecules}}$$
A molecular Bose condensate formed by SLOWLY ramping the magnetic field from just above ($a_s < 0$) to just below ($B - B_0 = -0.56G$) the resonance ($a_s > 0$).

\[
T = 0.19T_F \\
N_C = 0\% \\
T = 0.06T_F \\
N_C = 12\%
\]

The density profile of the free Fermi atoms is not shown. The molecular condensate has the same profile as an atomic BEC, except $M = 2m$.

Self-consistent solutions of the BdG equations for the local density of atoms $n(r)$ and the local order parameter $\tilde{\Delta}(r)$ in a harmonic trap. Normalized to values at center of trap.

Equation: $\tilde{\Delta}(r)$

Relative number of free Fermi atoms and Bosonic bound states as a function of position in trap.
In a trapped Fermi gas, we can form two-particle bound states (dimers) which are **Bosons** and hence they can Bose-condense, forming a **Fermi superfluid**.

In the **crossover**, we go from a region where Cooper pairs dominate to one where real molecules dominate. However, the **entire region** can be described by the **same BCS type formalism**, built on a condensate of pairs.

The **unbound or free Fermions** swim around in this condensate and are renormalized by the order parameter. In a trap, these **single particle excitations** have an energy gap $E_g$ and a spectrum that depends on both $\Delta(r)$ and $\mu$ in a **complicated way** (compared to usual BCS theory).
The holy grail has been to find evidence for a BCS Cooper pair condensate above the resonance, where $a_s < 0$. The bimodal profile shows evidence for the appearance of a Bose condensate of Cooper pairs. This experiment is done by ramping RAPIDLY from BCS region to BEC region so that real molecules do not have time to Bose-condense.

$B - B_0 = 0.55G$
$N_C = 1\%$ ---->

$B - B_0 = 0.12G$
$N_C = 10\%$ ---->


The density profile of the free atoms is not shown.
What about the single-particle **Fermi excitations** of these Fermi superfluids? Their spectrum can be used to probe the underlying **Bosonic condensate**. This is the hot area of research now in ultracold atom physics.

The sharp peaks at low energies come from the analogue of **Andreev states** at edge of trapped gas.

Graph of single-particle density of states $N(\omega)$ in a trap (in units of the trap frequency $\omega_0$).
Recently Grimm and coworkers at Innsbruck have used \textbf{rf-tunneling} of Fermi atoms into another atomic state. This type of measurement is the \textbf{analogue} of tunneling from a superconductor to a normal metal. It gives information about the \textbf{spectral density of the quasiparticle excitations} of the Fermi superfluid.

\[ T > T_C \]
\[ T << T_C \]

\textbf{BEC} \hspace{2cm} \textbf{Crossover region} \hspace{2cm} \textbf{BCS}
A standard calculation of the **tunneling current** gives it in terms of the single-particle Green’s functions:

\[
I_F(\omega) = \langle \hat{I}_F(\omega) \rangle = 2t_F^2 Im \int d\mathbf{r} d\mathbf{r}' \Pi_F(\mathbf{r}, \mathbf{r}', -\omega)
\]

where the effective detuning frequency is

\[
\omega \equiv \omega_L - \omega_a - \mu + \mu_a
\]

\[
\Pi_F(\mathbf{r}, \mathbf{r}', i\nu_n) \equiv -\int_0^\beta d\tau e^{i\nu_n \tau} \langle T_\tau \{ \Psi^\dagger_\alpha(\mathbf{r}, \tau) \Psi(\mathbf{r}, \tau) \Psi^\dagger_\beta(\mathbf{r}', \tau) \Psi_\beta(\mathbf{r}') \} \rangle
\]

\[
= \frac{1}{\beta} \sum_{i\omega_m} G_{11}(\mathbf{r}, \mathbf{r}', i\omega_m + i\nu_n) G_\alpha(\mathbf{r}', \mathbf{r}, i\omega_m).
\]

*Ohashi & Griffin, cond-mat/0410220*
Conclusions

- There is no **fundamental difference** between a molecular condensate in the **BEC limit** and a Cooper pair condensate in the **BCS limit**.

- The **single particle** Fermi excitations have the **BCS Bogoliubov spectrum** with an energy gap. However, this energy gap is now longer simply related to the order parameter even in the BCS region, but is due to **low energy Andreev states** localized near the edge of the trap.

- These single particle excitations can be **directly probed using rf-tunneling** into another atomic state.
With these *atomic superfluid* Fermi gases, we can study the effect of a pair condensate in a *direct way* compared to usual BCS superfluids, since the pairing interaction can be varied.

So far *s-wave* interactions have been mainly studied. However, *p-wave* and *d-wave* atomic Fermi superfluids are now being considered by theorists (*Ohashi*, cond-mat / 0410516 ; *Ho* and *Diener*, cond-mat / 0408468 ) and by experimentalists. *Stay tuned!*
My home city of Toronto, Canada in summer time!
2005 Banff Cold Atom Meeting: 10th Anniversary of Bose-Einstein Condensation

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