Quantum Imaging

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Resolution?

- Ability to separate details

XIX\textsuperscript{th} century: Lord Rayleigh

how accurately can I separate two objects in space?

Gemini north dome, Hawaii

Resolution is limited by spot size

Diffraction theory: resolution limited by the wavelength!
XXIst century: image sensor: diode arrays, CCD cameras, ...

There exist eigenmodes of the system (prolate spheroidal functions), $f_k(x)$ with eigenvalues $t_k$ (transmission coefficient).

The knowledge of these functions, together with $e(x)$, allows the ‘perfect’ reconstruction of the object.
Limits to resolution

Object plane $a(y)$

Imaging device with pupil

Image plane $e(x)$

Quality of the detectors: size, number of pixel, response,
Classical noise (vibrations, thermal noise,...)

Quantum nature of light (quantum noise)

Technical limits

Fundamental limit
Optical resolution vs. information extraction

Optical resolution

No a-priori information on the image: smallest details measurable.

- In many practical cases: the Rayleigh criteria.
- Crossing the standard quantum limits requires very multimode quantum light, i.e. many resources.
Information extraction

A lot of a-priori information: presence and/or modification of a given pattern.

- Quantum limit easier to reached: orders of magnitude smaller than the Rayleigh criteria.

- We will show that crossing the standard quantum limit requires a limited amount of resources.

Optical resolution vs. information extraction
Outline

- Single mode versus multimode light
- Quantum limits to resolution
- Few modes approach: the quantum laser pointer
- Many modes approach: multimode cavities
- Single mode versus multimode light
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- Few modes approach: the quantum laser pointer
- Many modes approach: multimode cavities
Modal decomposition of light

Paraxial approximation

A beam of light is the result of the excitation of an infinite set of harmonic oscillators. The electric field distribution can be expanded over a transverse mode basis:

- plane waves basis: very suitable for calculation

\[
E(\vec{r}, t) = \sum_{k} \alpha(k) e^{i(k \cdot \vec{r} - \omega(k) t)}
\]

However, for the propagation of a beam of light, we make several approximations:
- the light is monochromatic: \(\omega(k) = \omega_0\)
- the direction of propagation is well defined: \(k \parallel k_z\)

\[
E(\vec{\rho}, z, t) = \mathcal{E}(\vec{\rho}, z) e^{-i\omega_0(t - \frac{z}{c})} \text{ with } \vec{\rho} = (x, y)
\]

Where \(\mathcal{E}(\vec{\rho}, z)\) is the slowly varying envelope of the fields that satisfies the propagation equation in the vacuum, projected onto the polarisation axis:

\[
\frac{\partial^2 \mathcal{E}(\vec{\rho}, z)}{\partial x^2} + \frac{\partial^2 \mathcal{E}(\vec{\rho}, z)}{\partial y^2} + 2ik \frac{\partial \mathcal{E}(\vec{\rho}, z)}{\partial z} = 0
\]
Modal decomposition of light

Transverse modes basis

\[ \mathcal{E}(\vec{\rho}, z) \text{ can be expanded on a transverse modes basis such as:} \]
\[ \{ u_i(\vec{\rho}, z) \}_i \]

\[ \int u_i^*(z, \vec{\rho}) u_j(z, \vec{\rho}) d^2 \rho = \delta_{ij} \quad \text{orthonormality} \]
\[ \sum_i u_i^*(z, \vec{\rho}) u_i(z, \vec{\rho}') = \delta(\vec{\rho} - \vec{\rho}') \quad \text{completeness} \]

There is then a unique set of coefficient \( \alpha_i \) such as:

\[ \mathcal{E}(\vec{\rho}, z) = \sum_i \alpha_i u_i(\vec{\rho}, z) \]

- field amplitude in mode \( i \)
- shape of mode \( i \)

It contains all the image information

Remark:

As the modes have to satisfy the propagation equation, their knowledge at \( z=0 \) is enough.
Modal decomposition of light

Examples

- Pixel basis: 

\[ u_i(\vec{\rho}, z = 0) \approx \delta(\vec{\rho} - \vec{\rho}_i) \]

Advantages: very natural to describe random images, convenient for numerical simulation

Drawbacks: mode diffraction is very important, predicting the field shape under propagation is difficult

At \( z=0 \),

\[ E(\vec{\rho}) = \sum_i \alpha_i u_i(\vec{\rho}) \]

\[ E(\vec{\rho}, z) = \sum_i \alpha_i u_i(\vec{\rho}, z) \]
Gaussian modes

Gaussian modes basis: eigen modes of the propagation

These modes have a transverse shape that remain constant under propagation. They are adapted for light coming out of a cavity (such as laser beams).

Hermite-Gauss modes

Laguerre-Gauss modes
Single mode vs. multimode classical light?

Possible to compute the number of modes?

It depends on the choice of the basis!

For a field coming out of a cavity, one will naturally choose the Hermite Gauss or Laguerre Gauss basis.

Single mode basis

We have a given image: $\mathcal{E}(\vec{\rho}, z)$

We choose the first mode such as: $u_0(\vec{\rho}, z) = \frac{\mathcal{E}(\vec{\rho}, z)}{|\mathcal{E}(\vec{\rho}, z)|}$

It is always possible to choose the other modes to satisfy the completeness and orthonormality conditions: $\{u_i(\vec{\rho}, z)\}_i$

In that basis: $\mathcal{E}(\vec{\rho}, z) = \alpha_0 u_0(\vec{\rho}, z)$

No intrinsic definition of multimode at the classical level.
Quantum description of the field

Each mode is treated as a single harmonic oscillator

We associate to each mode a set of creation and annihilation operator

\[ u_i \rightarrow \hat{a}_i, \quad \hat{a}_i^\dagger \]

It allows to define the number of photon in each mode

The electric field operator

\[ \mathcal{E}(\vec{\rho}, z) = \sum_i \hat{a}_i(z) u_i(\vec{\rho}, z) \]

\[ \hat{a}_i = \langle \hat{a}_i \rangle + \delta \hat{a}_i \]

classical value \( \alpha_i \)

quantum fluctuations

\[ \mathcal{E}(\vec{\rho}, z) = \sum_i \alpha_i u_i(\vec{\rho}, z) \]
Detector: signal proportional to the number of photons

\[ \hat{N}(D) \propto \int_D \hat{E}^\dagger(\vec{\rho})\hat{E}(\vec{\rho}) d^2\rho \]

Signal and noise

The signal is given by the mean number of photon \( \langle \hat{N}(D) \rangle \)

The noise is the variance of the number of photons

\[ V(\hat{N}) = \Delta \hat{N}^2 = \langle \hat{N}(D)^2 \rangle - \langle \hat{N}(D) \rangle^2 \]
Single mode quantum field

Known classical image
\[ < \mathcal{E}(\vec{\rho}, z) > = \mathcal{E}(\vec{\rho}, z) = \alpha_0 u_0(\vec{\rho}, z) \]

Electric field operator
\[ \mathcal{E}(\vec{\rho}, z) = \sum_i \hat{a}_i(z) u_i(\vec{\rho}, z) \]

Annihilation operators
\[ < \hat{a}_0 > = \alpha_0 \]
\[ < \hat{a}_i >_{i \neq 0} = 0 \]

Single mode field
The field state in all the modes except the first one is a coherent vacuum

It then corresponds to the single mode quantum optics studied in the lecture of Hans Bacher.

It exists a proper definition of single mode at the quantum level

It is based on the quantum fluctuations
The same can be done for a statistical superposition of modes
- Single mode versus multimode light
- Quantum limits to resolution
- Few modes approach: the quantum laser pointer
- Many modes approach: multimode cavities
Light used in the experiment is single-mode coherent light.

\[ u_0(\vec{r}) \quad \text{Coherent state} \quad |\alpha_0\rangle \]
\[ u_1(\vec{r}) \quad \text{vacuum} \]
\[ \vdots \]
\[ u_n(\vec{r}) \quad \text{vacuum} \]

Measurement performed

Image carried by a Coherent state

\[ i_1(t) \]
\[ i_2(t) \]
\[ i_3(t) \]
\[ i_4(t) \]
Photon picture of coherent single mode light

Usual quantum optics description

Continuous wave regime (1mW ~ $10^{17}$ photons/s)

Photon number: **Poisson statistic** (also called white noise)

$$\Delta N = \sqrt{N}$$

Spatial quantum optics description

Random transverse distribution

Each detectors sees Poissonian noise

$$\Delta N_i = \sqrt{N_i}$$

Local Shot noise
Two pixels case

i₁ and i₂ not correlated:
Noise of the difference = noise of the sum

Smallest displacement detectable

Signal scales with \( \frac{N}{\sqrt{N}} \)
Noise scales with \( \frac{\Delta}{\sqrt{N}} \)

Example:

- Beam of 1 mW
- \( \Delta = 200 \mu \text{m} \)
- Integration time of 10 \( \mu \text{s} \)

\[ d_{lqs} = \frac{\Delta}{\sqrt{N}} \ll \lambda \]

\[ d_{lqs} = 5 \text{Å} \]

Dimensionless quantity
\[ \frac{-d}{\sqrt{\Delta}} = \frac{1}{\sqrt{N}} = 4.3 \times 10^{-6} \]
Relevant quantity:

\[ N(\rho) = N \text{ photons per m}^2 \]

Image characterisation:

\[ S_{sql} \approx \sqrt{2} \frac{L_{\text{var}}}{\sqrt{N(\rho)}} \]

\[ L_{\text{var}} \approx \frac{N(\rho)}{\| \nabla N(\rho) \|} \]
Standard quantum limit to resolution

1 dimension

$$d_{sql} = \frac{\Delta}{\sqrt{N}}$$

2 dimensions

$$S_{sql} \approx \sqrt{2} \frac{L_{var}}{\sqrt{N(\rho)}}$$

Image characteristic length

Shot noise or photon noise

Improve the sensitivity:
Increase the beam intensity
Reduce the photon noise
there exists eigenstates $f_n$ of the imaging device, with transmission $t_n$

$$E_{object} = f_n \quad \rightarrow \quad E_{image} = t_n f_n$$

$$E_{image} = \sum_n c_n f_n \quad \rightarrow \quad E_{object} = \sum_n \frac{c_n}{t_n} f_n$$

if the coefficients $c_n$ are perfectly known, object shape can be reconstructed **without limitation due to diffraction**
The knowledge of coefficients $C_n$ is not perfect. One can show that $(\Delta c_n)^2$ do not depend on $n$. $n$ small: $t_n \approx 1$; $n$ large: $t_n \ll 1$.

$n_{\text{max}}$ depends on Shannon number $dX/\lambda f$ of the set-up.

$E_{\text{object}} = \sum_n c_n f_n t_n$

$\delta x_{\text{min}} \approx X / n_{\text{max}}$

"Superresolution" very difficult in practice.
Outline

- Single mode versus multimode light
- Quantum limits to resolution
- Few modes approach: the quantum laser pointer
- Many modes approach: multimode cavities
Use single mode squeezed light?

Single-mode coherent light

Detector

Detection noise: shot noise

Photons randomly distributed in time and space
Use single mode squeezed light?

Single-mode squeezed light

Phonons ordered in time: detection noise below the shot noise

Phonons ordered in time but randomly distributed in space
Use single mode squeezed light?

- Single-mode squeezed light
- Photons ordered in time: detection noise below the shot noise
- Partial detection
Use single mode squeezed light?

Single-mode squeezed light

Partially detection

Photons ordered in time: detection noise below the shot noise

Partial detection is equivalent to a loss: no spatial order.

Partial measurement

No spatial squeezing
Multimode quantum light

Electric field operator

$$\hat{\mathcal{E}}(\vec{\rho}, z) = \sum_{i} \hat{a}_i(z) u_i(\vec{\rho}, z)$$

Known classical image

$$\langle \hat{\mathcal{E}}(\vec{\rho}, z) \rangle = \mathcal{E}(\vec{\rho}, z) = \alpha_0 u_0(\vec{\rho}, z)$$

Annihilation operators

$$\langle \hat{a}_0 \rangle = \alpha_0$$

$$\langle \hat{a}_i \rangle_{i \neq 0} = 0$$

Multimode light?

one of the other modes $u_1, u_2, u_3, \ldots$ is not in a coherent vacuum state

For instance: squeezed vacuum

Can be applied to any physical dimension.

Which mode for which measurement?
Linear measurement of an image

Pixel-like configuration

Image incident on a CCD camera

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Linear measurement

- **Intensity** on each detector: $N(D_i)$
- **Gain** on each detector: $\sigma_i$
- One measurement defined by:

$$N(\{\sigma_i\}) = \sum_i \sigma_i N(D_i)$$

Image is known
Measurement: a function of the gains
Difference measurement

Two identical signals from the light source

- zero mean signal

\[ \hat{N} = \hat{N}(D^+) - \hat{N}(D^-) \]

\[ \langle \hat{N} \rangle = 0 \]

With a classical field: \( \mathcal{E}(\rho) = \alpha_0 u_0(\rho) \) cancellation of the common mode noises

there is no noise in the measurement

However, with a quantum description: \( \hat{\mathcal{E}}(\rho) = \sum_i \hat{a}_i u_i(\rho) \)

there is quantum noise!
Noise in a difference measurement

Balanced detection

- Noise come from the vacuum port

Split detection

- Noise come from the flipped mode

\( u_0(r) \)
\( w(r) \)

\( \pi \) phase shift

intense beam

in phase

vacuum
Noise in a difference measurement

Image description

\[ \langle \hat{E}(\vec{p}) \rangle = \alpha_0 u_0(\vec{p}) \]

where \( u_0 \) is, for instance, a Gaussian mode

Origin of the noise

Noise originates from the flipped mode

\[ w(\vec{r}) = u_0(\vec{r}) \quad \text{if} \quad \vec{r} \in D^+ \]

\[ w(\vec{r}) = -u_0(\vec{r}) \quad \text{if} \quad \vec{r} \in D^- \]

Variance of the noise

\[ V(\hat{N}) = NV(\hat{X}_w^1) \]

Amplitude noise of mode \( w \)
Noise in a difference measurement

NOISE

\[ V(\tilde{N}) = NV(\tilde{X}_w^1) \]

- Noise on a difference measurement: from a single mode, the flipped mode.
- Reduce the noise in that measurement: necessary and sufficient to inject vacuum squeezing in that mode.

Transverse modes description

<table>
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<th>Modes</th>
<th>Spatially squeezed light</th>
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<tr>
<td>( u_0(\vec{r}) )</td>
<td>any state of mean value ( \alpha_0 )</td>
</tr>
<tr>
<td>( u_1(\vec{r}) = w(\vec{r}) )</td>
<td>squeezed vacuum</td>
</tr>
<tr>
<td>( u_2(\vec{r}) )</td>
<td>vacuum</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
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</table>
Noise in a general measurement

\[ I = i_+ - i_- \]

Variance of the noise

\[ V(\hat{N}) = NV(\hat{X}_w^1) \]

Transverse modes description

Same as for the differential measurement.
Noise in a general measurement

**General measurement**

\( \hat{N}(\{\sigma_j\}) = \sum_i \sigma_i \hat{N}(D_i) \)

Mean field mode: any shape

What is the detection mode?

**Detection mode**

It exists a detection mode \( w \) such as

\[ \text{if } \bar{\rho} \in D_i, w(\bar{\rho}) = \frac{1}{f} \sigma_i u_0(\bar{\rho}) \]

**Variance of the noise**

\[ V(\hat{N}) = f^2 NV(\hat{X}_w^1) \]
Application to the laser pointer

Measurement of a light beam with a quadrant detector

- Position / orientation of the beam
- Quantum limit of the measurements

Used in many physical apparatus such as: atomic force microscope, laser guided devices
Small displacements measurement

\[ I_x = (I_a + I_b) - (I_c + I_d) \]
\[ I_y = (I_a + I_c) - (I_b + I_d) \]

Beam shape

x flipped mode

y flipped mode
Small displacements measurement

Spatially squeezed laser beam

quadrant detector

$S_+ = SNL$

$S_{\text{horizontal}} = \text{squeeze}$

$S_{\text{vertical}} = \text{squeeze}$
Experimental implementation

Squeezed vacuum 2:

Mean field:

Squeezed vacuum 1:

"mode merger" displacement generator quadrant detector
Experimental Setup

Squeezed vacuum

Below threshold
Optical Parametric Amplifier

Transverse modes

“Cut and Paste” waveplate

Mode merger

Impedance matched cavity

Displacement generator

Physical system
Small displacements measurement

Impedance matched cavity

OPA 1

0°

waveplate 2

>95%

0°

>94%

180°

0°

0°

180°

waveplate 1

BS R=95%

Impedance matched cavity

OPA 2

180°

0°

0°

180°

Local oscillator
Spatial quantum noise

Relative vertical position

Vertical displacement

Horizontal displacement

Standard quantum limit

Limit with spatial squeezing

Simultaneous noise reduction: 2.6 dB vertical, 2.2 dB horizontal
Small displacements measurement

Oscillation at 4.5 MHz: mirror on a piezo-electric crystal.

Oscillation amplitude is linearly increased with time.

Signal measured

Coherent state

Spatially squeezed state
Outline

- Single mode versus multimode light
- Quantum limits to resolution
- Few modes approach: the quantum laser pointer
- Many modes approach: multimode cavities
The parametric process

Second order non-linearity

\( \chi^{(2)} \) medium: mix two different wavelengths.

Such as \( \omega_1 = \omega_2 + \omega_2 \)

Energy conservation

Parametric down conversion

Photons correlated in time and position

Used in many single photon experiments to create spatial entanglement
Single mode cavities

Cavity to increase the non linearity

Pump

$\chi^{(2)}$

signal

idler

Cavity select a spatial mode

The output is one Gaussian mode

The spatial order is lost!
Multimode cavities

Planar cavity

Spherical cavity
What can these cavities do?

Generate multimode quantum states:

- Generate multimode squeezed light:
  - Squeeze all the transverse modes simultaneously.
  - Generate spatial entanglement.
  ...

Noiseless amplification of images:

- The noise properties of the amplified image are better than what can be achieved with a classical amplifier.
Conclusion

- Quantum noise, and not diffraction, gives the ultimate limit to resolution

- The spatial dimension of light brings a lot of degree of freedom: many new quantum states are accessible.

- It is possible to improve several measurement performed on the same beam using appropriately designed spatially squeezed light.

- The future is toward improvement of practical apparatus (like optical resolution) on the one hand, and generation of highly multimode light on the other hand.
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<tr>
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<td>Magnus Hsu</td>
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Result of a very active collaboration between France and Australia. Cotutelle PhD students are welcome!
Some references

Theory on multimode light, classical and quantum


Information extraction and optical resolution

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