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Calorimetry of Bose–Einstein condensates

P B Blakie[1], **E Toth**[1] **and M J Davis**[2]

[1] Jack Dodd Centre for Photonics and Ultra-Cold Atoms, Department of Physics,
University of Otago, Dunedin, New Zealand
[2] ARC Centre of Excellence for Quantum-Atom Optics, School of Physical Sciences,
University of Queensland, Brisbane, QLD 4072, Australia

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Abstract

We outline a practical scheme for measuring the thermodynamic properties of a Bose–Einstein condensate as a function of internal energy. We propose using Bragg scattering and controlled trap manipulations to impart a precise amount of energy to a near-zero temperature condensate. After thermalization the temperature can be measured using standard techniques to determine the state equation $T(U, N, \omega)$. Our analysis accounts for interaction effects and the excitation of constants of motion which restrict the energy available for thermalization.

1. Introduction

The harmonically trapped Bose–Einstein condensate (BEC) system is well isolated from its environment and the thermal state can be characterized by the parameters of total atom number (N), internal energy (U) and trap potential frequency (ω). To date the internal energy of a Bose gas has been impractical to measure experimentally with a useful degree of accuracy. On the other hand, when a discernible thermal fraction is present the temperature can be quite accurately determined by absorption imaging after expansion [1]. For this reason, the temperature dependence of BEC thermodynamics, e.g. condensate fraction versus temperature, is quite well known, whereas the energy dependence has hardly been studied. The few experimental studies conducted have suffered from large uncertainties such that any form of quantitative comparison with theory was not possible [2]. More recent studies have used an increase in temperature to signify the imparting of energy [3, 4], but have not considered the quantitative relationship between these two quantities.

Knowledge of the energy dependence of BEC thermodynamics is of wide spread interest. The λ-transition in He was so named according to the peculiar shape in the specific heat capacity of the system [5]. Additionally a detailed description of energy dependence would be useful for discriminating between finite temperature theories of ultra-cold Bose gases. For the case of degenerate Fermi gases, a heat capacity measurement has been made by the Duke group [6] by manipulating the trap potential in a manner similar to what we consider here for Bose

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gases. Additionally, recent work by the Heidelberg group [7] has examined a precise method for measuring the temperature, and used this to confirm the deviation of the heat capacity of a Bose gas from that of a classical gas for a constant background noise source. However, their input heating rate was unknown, so that they were unable to quantify the heat capacity. The MIT group [8, 9] have measured heating in a BEC by stirring it with a blue-detuned focused light beam, and used those measurements to distinguish between theories for drag forces.

Here we propose using two mechanisms for transferring a precise amount of energy to a BEC at near-zero temperature to establish the relationship between energy and temperature. For making calorimetric measurements, one ideally would like a well-defined reservoir to transfer heat to the system of interest. The isolation of ultra-cold atom experiments makes such an approach impractical; however, an irreversible work process, such as that done by a spinning paddle wheel in a fluid, is a convenient method for transferring energy into these systems without changing the external constraints.

Our main concern in this proposal is to develop and analyse precise ways of imparting energy to the system, and to characterize the portion of this energy that is irreversible. While our analysis here focuses upon Bragg scattering, and expansion from a trap, one could envisage doing this with other methods, e.g. general perturbations of the trapping potential or stirring with a focused light field. The main requirement is that the energy transfer is accurately calculable.

2. Precise energy transfer

2.1. Overview of proposal

In the next subsections, we analyse two methods for precisely transferring energy ($\mathcal{E}_{\mathrm{trans}}$) to the system to be rethermalized. In section 2.3, we consider the use of Bragg scattering, and in section 2.4 we consider the sudden expansion from a harmonic trap. Having added this energy, the final temperature of the system (after it has returned to equilibrium) can be accurately measured. Knowledge of the irreversible work done on the system, and the rethermalized temperature establishes an equation of state relationship of the form $T(U)$, where T is the temperature and U is the internal energy (relative to the energy of the $T = 0$ ground state). An important consideration in equating U to $\mathcal{E}_{\mathrm{trans}}$ is that $\mathcal{E}_{\mathrm{trans}}$ must only consist of the irreversible work done on the system. In particular, energy transferred to the Kohn mode (of harmonically trapped gases) needs to be excluded (as we discuss below).

2.2. Zero-temperature formalism

We consider our initial system to be a Bose gas at zero temperature, where the condensate is essentially pure. The condensate orbital satisfies the time-independent Gross–Pitaevskii equation

$$\mu\Psi_g = \hat{H}_{\mathrm{sp}}\Psi_g + NU_0|\Psi_g|^2\Psi_g, \tag{1}$$

where $\hat{H}_{\mathrm{sp}} = \hat{p}^2/2m + V_{\mathrm{H}}(\mathbf{x})$ is the single particle Hamiltonian with

$$V_{\mathrm{H}}(\mathbf{x}) = \frac{1}{2}m\sum_{j=1}^{3}\omega_j^2 x_j^2, \tag{2}$$

the harmonic trapping potential, and $\{\omega_1, \omega_2, \omega_3\}$ are the trap frequencies along the coordinate directions. The quantity μ is the chemical potential, N is the number of particles in the initial (pure) condensate, and $U_0 = 4\pi a\hbar^2/m$ is the interaction strength, where a is the s-wave scattering length.

The energy of the ground state is given by the energy functional

$$\mathcal{E}_g = E[\Psi_g] = \int \mathrm{d}^3\mathbf{x}\, N\Psi_g^* \left[\hat{H}_{\mathrm{sp}} + \frac{NU_0}{2}|\Psi_g|^2\right]\Psi_g. \tag{3}$$

Many-body corrections to the ground state will in general be important; however, our interest here lies in understanding how much energy is transferred to the system rather than the absolute energy, for which the energy functional will suffice.

On several occasions we will have cause to make use of the Thomas–Fermi approximate solution to the Gross–Pitaevskii equation (1)

$$|\Psi_g|^2 \approx |\Psi_{\mathrm{TF}}(\mathbf{x})|^2 = \left\{\begin{matrix} \frac{\mu_{\mathrm{TF}} - V_{\mathrm{H}}(\mathbf{x})}{NU_0}, & V_{\mathrm{H}} < \mu_{\mathrm{TF}}, \\ 0, & \text{elsewhere} \end{matrix}\right\}, \tag{4}$$

with the chemical potential determined by

$$\mu_{\mathrm{TF}} = \frac{\hbar\bar{\omega}}{2}\left(15Na\sqrt{\frac{m\bar{\omega}}{\hbar}}\right)^{2/5}, \tag{5}$$

where $\bar{\omega} = (\omega_1\omega_2\omega_3)^{1/3}$ (e.g. see [10]). This approximation is found by neglecting the kinetic term in the Gross–Pitaevskii equation, which is usually much smaller than the potential and interaction contributions. Comparisons with experiments have shown the Thomas–Fermi approximation to be a good description of $T = 0$ condensates (improving in accuracy as the number of particles in the condensate increases, e.g. see [10]).

The Thomas–Fermi ground state has an energy of

$$\mathcal{E}_g = \tfrac{5}{7}N\mu_{\mathrm{TF}}, \tag{6}$$

calculated from equation (3), with the kinetic term neglected.

2.3. Method I: energy transfer by Bragg scattering

Our first scheme for imparting energy uses the well-understood process of Bragg scattering [11–13] that is routinely used in labs for manipulating BECs. We consider the situation where a Bragg pulse is used to first-order Bragg scatter a fraction α of the condensate (at rest) to momentum state $\hbar\mathbf{b}$ (where $\mathbf{b}$ is the reciprocal lattice vector of the Bragg potential). Note that the harmonic trap remains on during the Bragg scattering and the subsequent dynamics of the system as it rethermalizes. We also assume that the duration and intensity of the Bragg potential are chosen so that all other orders of scattering can be neglected, yet the scattering can be considered approximately instantaneous on the timescale of condensate evolution (e.g. see [14]). The matter wave field at the conclusion of the scattering is then given by

$$\Psi_i(\mathbf{x}) = \sqrt{1-\alpha}\,\Psi_g(\mathbf{x}) + \sqrt{\alpha}\,\mathrm{e}^{\mathrm{i}\mathbf{b}\cdot\mathbf{x}}\Psi_g(\mathbf{x}), \tag{7}$$

where we have assumed that the size of the condensate is large compared to $1/\mathbf{b}$, so that the wave packets centred at momenta $\mathbf{0}$ and $\hbar\mathbf{b}$ are orthogonal.

2.3.1. Kohn mode. The initial state (7) has a momentum expectation of $\mathbf{p}_i = \alpha\hbar\mathbf{b}$ per particle and according to Kohn's theorem [15] this will lead to an undamped dipole oscillation in the harmonic external potential. Transforming to the (non-inertial) time-dependent centre-of-mass frame of reference, this oscillation can be removed, while leaving the Hamiltonian

Table 1. Energy scales per particle for typical experimental parameters. Each case considers a 10^6 atom condensate in an isotropic 50 Hz harmonic trap. We have taken the usual scattering lengths for each atom [10] and have taken b to be that for counter-propagating light fields of wavelength $\lambda = 589$ nm and $\lambda = 789$ nm for the sodium and rubidium cases respectively (i.e. $b = 4\pi/\lambda$).

Atom	$\hbar\omega_b$	$2\mathcal{E}^0_{\text{int}}/N$
^{23}Na	6.6×10^{-29} J $\approx 4.8\ \mu$K	4.3×10^{-31} J $\approx 0.031\ \mu$K
^{87}Rb	9.8×10^{-30} J $\approx 0.71\ \mu$K	7.3×10^{-31} J $\approx 0.053\ \mu$K

for the system unchanged (see [15]). In making this transformation an amount of energy corresponding to the energy of the dipole oscillation, i.e.

$$E_D = Np_i^2/2m = \alpha^2 N\hbar\omega_b, \tag{8}$$

is removed, where $\omega_b = \hbar b^2/2m$ is the Bragg recoil frequency. Because this energy is locked into the centre of mass oscillation it is not available for rethermalization.

However, if the motion of the scattered atoms enters into a sufficiently anharmonic region of the trapping potential, then the Kohn mode energy will be available to rethermalize. We expect this to be strongly dependent on the manner in which the harmonic potential is made, but this effect should be clearly observable as a decay in the centre-of-mass (COM) oscillation of the system.

2.3.2. Transferred energy. The energy transferred to the condensate in the lab frame by the Bragg scattering is calculated in appendix A.1. After subtracting the energy locked into the Kohn mode we obtain that transferred energy available for rethermalization is

$$\mathcal{E}_{\text{trans}} \simeq N\hbar\omega_b(\alpha - \alpha^2) + 2\mathcal{E}^0_{\text{int}}(\alpha - \alpha^2), \tag{9}$$

where $\mathcal{E}^0_{\text{int}} = (N^2U_0/2)\int d^3\mathbf{x}|\Psi_g(\mathbf{x})|^4$ is the interaction energy of the ground state. In the term proportional to $\hbar\omega_b$ in equation (9), the α contribution arises from the transfer of kinetic energy to the particles, where as the $-\alpha^2$ part accounts for the energy locked into the Kohn mode. The term proportional to $2\mathcal{E}^0_{\text{int}}$ describes the additional energy arising from interactions due to the creation of a coherent density fluctuation in the system. We have taken this term to lowest order in the small parameter λ/L, where $\lambda = 2\pi/b$ is the wavelength of the density fluctuation and L is the size of the condensate. For more details we refer the reader to appendix A.1. The size of these kinetic and interaction contributions to the energy are compared in table 1. We note that for both cases considered the interaction contribution is $\lesssim 7\%$ the kinetic contribution. So for many cases ignoring the interaction term in equation (9) will be a good first approximation.

We note that the maximum transfer of energy occurs when $\alpha = 1/2$, since the energy available for rethermalization is proportional to the momentum spread in the initial state (7) which is maximized for 50% scattering. For $\alpha > 1/2$ the energy transferred by the Bragg scattering is increasingly locked into the Kohn mode.

2.4. Method II: energy transfer by expansion from trap

Our second method of energy transfer is to suddenly turn off the harmonic trapping potential (of initial frequencies $\{\omega_j\}$) for a period of time t_{on}, allowing the condensate to expand, before the potential is reinstated (with final frequencies $\{\omega'_j\}$).

To quantify the energy transfer in this process we need to consider the condensate dynamics after trap release. Castin and Dum have shown that such a system will undergo a *self-similar expansion* [16]. In particular, if the trapping frequencies are time dependent, then the condensate density remains as a Thomas–Fermi profile, but with time-dependent widths that evolve according to

$$R_j(t) = \lambda_j(t) R_j(0), \tag{10}$$

where the equations of motion for the λ_j are

$$\ddot{\lambda}_j = \frac{\omega_j^2(0)}{\lambda_j \lambda_1 \lambda_2 \lambda_3} - \omega_j^2(t)\lambda_j, \tag{11}$$

with $\lambda_j(t=0) = 1$.

2.4.1. Kohn mode. When the trap is turned off at $t = 0$ the condensate begins to expand and it will also fall a distance $d = a_g t^2/2$ and acquire a velocity $v = a_g t$, where a_g is the acceleration due to gravity. At time t_{on}, when the trap is restored, d and v will manifest themselves as energy locked into the dipole mode[3]. We note that, depending on how the harmonic trap is produced, the trap centre may also change if the trap frequencies of the final trap are different to those of the initial trap. For this reason we do not give explicit expressions for the dipole energy here.

2.4.2. Transferred energy. Using the results of appendix A.2, we find that the energy transferred and available for thermalization is

$$\mathcal{E}_{\text{trans}} = \frac{N\mu_{\text{TF}}}{7}\left(2 - 5\bar{\gamma}^{6/5} + \sum_{j=1}^{3} \gamma_j^2 \lambda_j^2(t_{\text{on}})\right), \tag{12}$$

where $\gamma_j = \omega'_j/\omega_j$, $\bar{\gamma} = \sqrt[3]{\gamma_1\gamma_2\gamma_3}$ and μ_{TF} is the Thomas–Fermi chemical potential of the initial condensate. Unlike the Bragg case, the transferred energy is unbounded since the condensate can be allowed to expand for arbitrarily long time periods.

2.4.3. Analytic solution. While the energy transfer generally requires us to solve the ordinary differential equations (11), an approximate solution exists for the case of an elongated (cigar) trap. Here the parameters $\lambda_\perp$ and λ_z specify the system at time t. Defining $\epsilon = \omega_\perp/\omega_z$ and $\tau = \omega_\perp t$ we can find an approximate solution [16]

$$\lambda_\perp = \sqrt{1+\tau^2}, \tag{13}$$

$$\lambda_z = 1 + \epsilon^2\left[\tau \tan^{-1}\tau - \ln\sqrt{1+\tau^2}\right] + O(\epsilon^4). \tag{14}$$

Thus the energy available for thermalization given the same initial and final trap parameters is

$$\mathcal{E}_{\text{trans}} = \frac{2N\mu_{\text{TF}}}{7}\left(\tau_{\text{on}}^2 + \epsilon^2\left[\tau_{\text{on}} \tan^{-1}\tau_{\text{on}} - \ln\sqrt{1+\tau_{\text{on}}^2}\right] + O(\epsilon^4)\right), \tag{15}$$

with $\tau_{\text{on}} = \omega_\perp t_{\text{on}}$.

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[3] We note that the pure expansion is symmetric and does not couple to the dipole mode. Any energy locked into the dipole mode will be due solely to the effect gravity and changes in the trap equilibrium position.

3. Application of calorimetry

We now consider the application of our calorimetry scheme to an ideal trapped Bose gas with critical temperature given by $T_c = \frac{\hbar\bar{\omega}}{k_B}[N/\zeta(3)]^{1/3}$. For $T < T_c$, we have that the energy of this gas is given by

$$U(T) = 3Nk_B \frac{\zeta(4)}{\zeta(3)} \frac{T^4}{T_c^3}, \tag{16}$$

where $\zeta(\alpha) = \sum_{n=1}^{\infty} n^{-\alpha}$ (e.g. see [17]).

3.1. Bragg limitations

In order for the Bragg scheme to be capable of probing up to the transition region, we require that maximum energy transfer is greater than the energy content of the gas at the critical point (i.e. $\mathcal{E}_{\text{trans}}(\alpha = 1/2) > U(T_c)$). Evaluating this inequality sets the following constraint on the system size and trap

$$\hbar\bar{\omega}\sqrt[3]{N} < \frac{\hbar\omega_b}{12} \frac{\zeta(3)^{4/3}}{\zeta(4)}. \tag{17}$$

Thus for N or $\bar{\omega}$ too large, first-order Bragg will not provide sufficient energy to heat the BEC from $T = 0$ to $T = T_c$. For example, 150×10^3 rubidium-87 atoms in a 25 Hz trap is about at this threshold. For sodium-23, the higher kinetic energy of the Bragg scattering ($\hbar\omega_b$) allows larger/tighter systems to be used. One could circumvent this limitation by using second-order Bragg scattering to impart a larger amount of energy.

We note that given the high recoil temperatures (relative to usual T_c values) associated with the light used to Bragg scatter it may seem surprising that it is not always possible to Bragg scatter sufficient energy to take the system to the critical point. However, it should be kept in mind that the recoil temperature is defined for an atom with a mean energy equal to the recoil energy in each degree of freedom, whereas for our Bragg scattering process this energy must be shared between the 6 degrees of freedom of the atom (i.e. 3 kinetic and 3 potential), and at most only 50% of the total energy transferred is available for rethermalization.

3.2. Error analysis of Bragg energy transfer

Error in the transferred energy will likely be dominated by shot-to-shot variation in the number of atoms in the initial condensate and error in the fraction of scattered atoms. Here we denote these errors as ΔN and $\Delta\alpha$ respectively and account for their effect on our ability to know the precise amount of energy transferred to the system. Linearizing equation (9), we find that the energy transferred is affected by these quantities according to

$$\Delta\mathcal{E}_{\text{trans}} = \hbar\omega_b[(\alpha - \alpha^2)\Delta N + N(1 - 2\alpha)\Delta\alpha], \tag{18}$$

where we have neglected the contributions of the interaction term since it is typically an order of magnitude smaller. In figure 1 we consider the effect of these errors in the use of our scheme to determine the heat capacity.

Additional error will arise because the initial condensate temperature is nonzero; however, since the energy is a rapidly increasing function of T^4, this error is typically several orders of magnitude smaller than the errors due to uncertainty in atom number and scattered fraction.

In figure 1, we also compare $U(T)$ for the interacting versus ideal system. The interacting properties are calculated using the Hartree–Fock–Bogoliubov theory in the Popov

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[4] For the ideal Bose gas $U \sim T^4$.

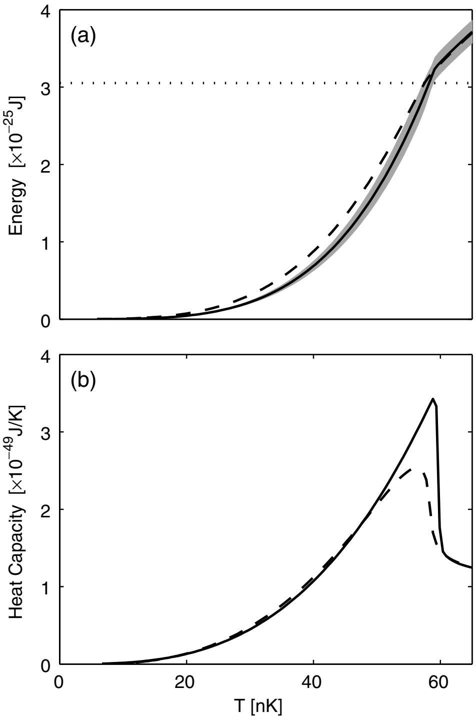

Figure 1. (a) Internal energy (U) versus temperature. (b) The specific heat capacity ($\Delta U/\Delta T$). Ideal gas (solid line), Hartree–Fock–Bogoliubov–Popov calculation of interacting gas (dashed). Energy corresponding to $\alpha = 0.5$ is shown as horizontal dotted line in (a) and grey region indicates the uncertainty in the transferred energy for the ideal case given that there is a 5% uncertainty in the total number of atoms (i.e. $\Delta N = 0.05 \times N$), the fraction scattered is accurate to the 5% level (i.e. $\Delta\alpha = 0.05 \times \alpha$). System parameters: 1.5×10^5Rb atoms in an isotropic 25 Hz trap.

approximation (e.g. see [18–22]). We see that for $T < T_c \sim 60$ nK the dependence of U on T for the interacting system is noticeably distinguishable from the ideal gas (i.e. beyond the limits of the error in energy transfer). This suggests that interaction effects could be experimentally measured using this technique if sufficiently good reproducibility of initial condensate number and Bragg scattering precision can be obtained.

3.3. Error analysis of energy transfer by trap expansion

As discussed in section 2.4.2 imparting energy by trap expansion has no upper bound to the amount of transferred energy. We would expect that the shot to shot variation in atom numbers will dominate the error budget; however, there may be other important considerations relating to the particular way the trap is produced.

For the analytic case given in section 2.4.3, we examine the sensitivity of the energy transferred to variations in atom number (ΔN) and errors in the axial ($\Delta\omega_z$) and radial ($\Delta\omega_\perp$) trapping frequencies. By linearizing equation (15) we find

$$\Delta\mathcal{E}_{\text{trans}} = \frac{7}{5}\mathcal{E}_{\text{trans}}\frac{\Delta N}{N} + \left(\frac{4}{7}N\mu_{\text{TF}}\tau_{\text{on}}^2 - \frac{8}{5}\mathcal{E}_{\text{trans}}\right)\frac{\Delta\omega_z}{\omega_z} + \left(\frac{2}{7}N\mu_{\text{TF}}\epsilon^2\tau_{\text{on}}\tan^{-1}\tau_{\text{on}} + \frac{14}{5}\mathcal{E}_{\text{trans}}\right)\frac{\Delta\omega_\perp}{\omega_\perp}. \tag{19}$$

Assuming that the dominant uncertainty is atom number variation, we see that the relative uncertainty in transferred energy is roughly proportional to the relative uncertainty in atom number. For the general case given in equation (12), the error associated with trap uncertainties requires a numerical integration of the differential equation (11). However, the (likely dominant) error associated with number uncertainty is still given by the first term in equation (19).

4. Conclusions

We have presented two practical schemes for performing calorimetry on a Bose–Einstein condensate system. It is clear from our results that reasonably accurate calorimetry measurements could be made using Bragg scattering or by controlled expansion from a confining potential. We have characterized the sensitivity of these methods to typical experimental uncertainties in atom number and have also shown that it should be feasible to measure interaction effects on the thermal properties of a Bose gas.

Our scheme also presents a rather well-defined initial condition for studying non-equilibrium dynamics. There is considerable interest in the dynamics of the thermalization, as we expect there will be a crossover from coherent to incoherent dynamics. This topic is of significant current interest (e.g. see [23–31]) and will be the subject of future work using classical field methods (e.g. [32, 33]).

Appendix. Energy calculations

A.1. Bragg transferred energy

Here we calculate the total energy transferred to the condensate by Bragg scattering in the lab frame. This is found by evaluating $\mathcal{E}_{\text{trans}} = E[\Psi_i] - E[\Psi_g]$. Assuming that the original and scattered wave packets are well separated in momentum space, we can make the approximations

$$\int \mathrm{d}^3\mathbf{x}|\Psi_g|^2\,\mathrm{e}^{\mathrm{i}\mathbf{b}\cdot\mathbf{x}} \approx 0, \tag{A.1}$$

$$\int \mathrm{d}^3\mathbf{x}|\Psi_g|^4\,\mathrm{e}^{\mathrm{i}\mathbf{b}\cdot\mathbf{x}} \approx 0, \tag{A.2}$$

$$\int \mathrm{d}^3\mathbf{x}|\Psi_g|^2\,\mathrm{e}^{2\mathrm{i}\mathbf{b}\cdot\mathbf{x}} \approx 0. \tag{A.3}$$

These integrals are all of order λ/L where L is the spatial size of the condensate and $\lambda = 2\pi/b$ is the wavelength of the Bragg-induced density modulation. In experiments typical values are $\lambda/L \sim 1/50$, so these approximations are well satisfied and higher order terms can be ignored.

With the above approximations we obtain

$$E[\Psi_i] = \mathcal{E}^0_{\text{sp}} + \alpha N\hbar\omega_b + (1 + 2\alpha - 2\alpha^2)\mathcal{E}^0_{\text{int}}, \tag{A.4}$$

where

$$\mathcal{E}^0_{\text{sp}} = N\int \mathrm{d}^3\mathbf{x}\Psi_g^*\hat{H}_{\text{sp}}\Psi_g, \tag{A.5}$$

$$\mathcal{E}^0_{\text{int}} = \frac{N^2U_0}{2}\int \mathrm{d}^3\mathbf{x}|\Psi_g|^4, \tag{A.6}$$

are the single-particle and interaction energies of the ground state respectively, with $\mathcal{E}_g = \mathcal{E}_{\rm sp}^0 + \mathcal{E}_{\rm int}^0$. Thus the transferred energy is given by

$$\mathcal{E}_{\rm Bragg} = \alpha N \hbar \omega_b + 2(\alpha - \alpha^2)\mathcal{E}_{\rm int}^0. \tag{A.7}$$

The Thomas–Fermi approximation to the ground state (4) gives

$$\mathcal{E}_{\rm int}^0 = \tfrac{2}{7} N \mu_{\rm TF}. \tag{A.8}$$

A.2. Trap expansion transferred energy

Having solved equations (11), the condensate density is then given by

$$|\Psi(\mathbf{x},t)|^2 = \frac{\mu_{\rm TF} - \sum_j \frac{1}{2} m\omega_j(0)^2 x_j^2 / \lambda_j^2(t)}{N U_0 \lambda_1(t)\lambda_2(t)\lambda_3(t)}. \tag{A.9}$$

A.2.1. Expansion. We envisage turning the trap off suddenly. In the Thomas–Fermi approximation, for the ground state, the kinetic energy is negligible meaning that the system only has interaction energy given by the generalized form of equation (A.6)

$$\mathcal{E}_{\rm int}(t) = \frac{N^2 U_0}{2} \int {\rm d}^3\mathbf{x} |\Psi(\mathbf{x},t)|^4. \tag{A.10}$$

The condensate begins to expand: the interaction energy is converted to kinetic energy ($\mathcal{E}_{\rm kin}$). The energy balance is determined by $\mathcal{E}_{\rm kin}(t) = \mathcal{E}_{\rm int}(0) - \mathcal{E}_{\rm int}(t)$, and using the time-dependent density of equation (A.9) we find

$$\mathcal{E}_{\rm int}(t) = \frac{2N\mu_{\rm TF}}{7\lambda_1(t)\lambda_2(t)\lambda_3(t)}. \tag{A.11}$$

Now consider turning the trap back on suddenly at time $t = t_{\rm on}$, with some possibly different trap frequencies $\{\omega_j'\}$. Defining the ratio of new to old trapping frequencies as $\gamma_j = \omega_j'/\omega_j$, the potential energy at the instant the new potential is turned on is given by

$$\mathcal{E}_{\rm pot}' = \int {\rm d}^3\mathbf{x} V_{\rm H}'(\mathbf{x}) |\Psi(\mathbf{x}, t_{\rm on})|^2, \tag{A.12}$$

$$= \frac{N\mu_{\rm TF}}{7} \sum_{j=1}^{3} \gamma_j^2 \lambda_j^2(t_{\rm on}). \tag{A.13}$$

and the total energy of the system will be the sum of the interaction energy at $t = 0$ and the potential energy at $t = t_{\rm on}$, i.e.

$$\mathcal{E}_{\rm tot}' = \frac{N\mu_{\rm TF}}{7} \left(2 + \sum_{j=1}^{3} \gamma_j^2 \lambda_j^2(t_{\rm on}) \right). \tag{A.14}$$

The using the Thomas–Fermi result in equation (6) we find the approximate ground-state energy in the final trapping potential

$$\mathcal{E}_g' = \frac{5N\mu_{\rm TF}\bar{\gamma}^{6/5}}{7} \tag{A.15}$$

where we have defined $\bar{\gamma}$ as the geometric mean of the γ_j. Thus the energy added by the sudden release and re-application of the harmonic trapping potential is

$$\mathcal{E}_{\rm exp} = \mathcal{E}_{\rm tot}' - \mathcal{E}_g', \tag{A.16}$$

$$= \frac{N\mu_{\rm TF}}{7} \left(2 - 5\bar{\gamma}^{6/5} + \sum_{j=1}^{3} \gamma_j^2 \lambda_j^2(t_{\rm on}) \right). \tag{A.17}$$

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