Qudit Entanglement[*]

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Abstract

We consider the separability of various joint states of D-dimensional quantum systems, which we call "qudits." We derive two main results: (i) the separability condition for a two-qudit state that is a mixture of the maximally mixed state and a maximally entangled state; (ii) lower and upper bounds on the size of the neighborhood of separable states surrounding the maximally mixed state for N qudits.

1 Introduction

One of the distinguishing features of quantum mechanics, not found in classical physics, is the possibility of entanglement between subsystems. It lies at the core of many applications in the emerging field of quantum information science [1], such as quantum teleportation [2] and quantum error correction [3, 4]. Entanglement is a distinctly quantum-mechanical correlation between subsystems, which cannot be created by actions on each subsystem separately; moreover, correlations between subsystem measurements on an entangled composite system cannot be explained in terms of correlations between local classical properties inherent in the subsystems. Thus one often says that an entangled composite system is nonseparable. Formally, the state of a composite system, pure or mixed, is *separable* if the state has an ensemble decomposition in terms of product states. A separable state has no quantum entanglement, and a nonseparable state is entangled. Though the nonclassical nature of quantum entanglement has been recognized for many years [5, 6], only recently has

considerable attention been focused on trying to understand and characterize its properties precisely.

This paper focuses on the question of whether various joint quantum states of D-dimensional quantum systems are entangled. For convenience, we call a D-dimensional quantum system a "qudit," by analogy with the name "qubit" for $D = 2$ and "qutrit" for $D = 3$. We now have a general method for quantifying the degree of entanglement of a pair of qubits [7], and we have a criterion, the partial transposition condition of Peres [8], which determines whether a general state of two qubits is entangled and whether a general state of a qubit and a qutrit is entangled [9]. The partial-transposition condition fails, however, to provide a criterion for entanglement in other cases, where the constituents have higher Hilbert-space dimensions [10, 11] or where there are more than two constituents. Indeed, at present there is no general criterion for determining whether the joint state of N qudits is entangled, nor is there any general way to quantify the degree of entanglement if such a state is known to be entangled.

In Sect. 2 we review an operator representation of qudit states, which is applied in Sect. 3, where we consider states of two qudits that are a mixture of the maximally mixed state and a maximally entangled state. We show that such states are separable if and only if the probability for the maximally entangled state in the mixture does not exceed $1/(1+D)$. This result was obtained by Horodecki and Horodecki [12], and a more general result, of which this is a special case, was obtained by Vidal and Tarrach [13]. In Sect. 4 we consider the separability of mixed states of N qudits near the maximally mixed state. We find both lower and upper bounds on the size of the neighborhood of separable states around the maximally mixed state. Our results generalize and extend the results obtained by Braunstein *et al.* for qubits [14] and by Caves and Milburn for qutrits [15]. Before tackling the upper and lower bounds, we present, in Sect. 4.1, various mathematical results which are used to obtain the lower bound, but which might prove useful in other contexts as well.

2   Operator representation of qudit states

In this section we review an operator representation of qudit states, analogous to the Pauli, or Bloch-sphere, representation for qubits. We begin with the set of Hermitian generators of $SU(D)$; the generators, denoted by λ_j, are labeled by a Roman index taken from the middle of the alphabet, which takes on values $j = 1, \ldots, D^2 - 1$. We represent the generators in an orthonormal basis $|a\rangle$, labeled by a Roman letter taken from the beginning of the alphabet, which takes on values $a = 1, \ldots, D$. With these conventions the generators are given by

$$j = 1, \ldots, D-1 :$$
$$\lambda_j = \Gamma_a \equiv \frac{1}{\sqrt{a(a-1)}} \left(\sum_{b=1}^{a-1} |b\rangle\langle b| - (a-1)|a\rangle\langle a| \right) , \quad 2 \le a \le D , \quad (1)$$
$$j = D, \ldots, (D+2)(D-1)/2 :$$
$$\lambda_j = \Gamma_{ab}^{(+)} \equiv \frac{1}{\sqrt{2}} \left(|a\rangle\langle b| + |b\rangle\langle a| \right) , \quad 1 \le a < b \le D , \quad (2)$$
$$j = D(D+2)/2, \ldots, D^2 - 1 :$$
$$\lambda_j = \Gamma_{ab}^{(-)} \equiv \frac{-i}{\sqrt{2}} \left(|a\rangle\langle b| - |b\rangle\langle a| \right) , \quad 1 \le a < b \le D . \quad (3)$$

In Eqs. (2) and (3), the Roman index j stands for the pair of Roman indices, ab, whereas in Eq. (1), it stands for a single Roman index a. The generators are traceless and satisfy

$$\lambda_j \lambda_k = \frac{1}{D}\delta_{jk} + d_{jkl}\lambda_l + if_{jkl}\lambda_l \ . \tag{4}$$

Here and wherever it is convenient throughout this paper, we use the summation convention to indicate a sum on repeated indices. The coefficients f_{jkl}, the structure constants of the Lie group $\mathrm{SU}(D)$, are given by the commutators of the generators and are completely antisymmetric in the three indices. The coefficients d_{jkl} are given by the anti-commutators of the generators and are completely symmetric.

By supplementing the $D^2 - 1$ generators with the operator

$$\lambda_0 \equiv \frac{1}{\sqrt{D}}I \ , \tag{5}$$

where I is the unit operator, we obtain a Hermitian operator basis for the space of linear operators in the qudit Hilbert space. This is an orthonormal basis, satisfying

$$\mathrm{tr}(\lambda_\alpha \lambda_\beta) = \delta_{\alpha\beta} \ . \tag{6}$$

Here the Greek indices take on the values $0, \ldots, D^2 - 1$; throughout this paper, Greek indices take on D^2 or more values. Using this orthonormality relation, we can invert Eqs. (1)–(3) to give

$$|a\rangle\langle a| = \frac{I}{D} + \frac{1}{\sqrt{a(a-1)}}\left(-(a-1)\Gamma_a + \sum_{b=a+1}^{D}\Gamma_b\right) \ , \tag{7}$$

$$|a\rangle\langle b| = \frac{1}{\sqrt{2}}(\Gamma_{ab}^{(+)} + i\Gamma_{ab}^{(-)}) \ , \quad 1 \le a < b \le D \ , \tag{8}$$

$$|b\rangle\langle a| = \frac{1}{\sqrt{2}}(\Gamma_{ab}^{(+)} - i\Gamma_{ab}^{(-)}) \ , \quad 1 \le a < b \le D \ . \tag{9}$$

Any qudit density operator can be expanded uniquely as

$$\rho = \frac{1}{D}c_\alpha \lambda_\alpha \ , \tag{10}$$

where the (real) expansion coefficients are given by

$$c_\alpha = D\mathrm{tr}(\rho\lambda_\alpha) \ . \tag{11}$$

Normalization implies that $c_0 = \sqrt{D}$, so the density operator takes the form

$$\rho = \frac{1}{D}\left(I + c_j\lambda_j\right) = \frac{1}{D}(I + \vec{c} \cdot \vec{\lambda}) \ . \tag{12}$$

Here $\vec{c} = c_j\vec{e}_j$ can be regarded as a vector in a (D^2-1)-dimensional real vector space, spanned by the orthonormal basis $\vec{e}_j$, and $\vec{\lambda} = \lambda_j\vec{e}_j$ is an operator-valued vector. If $\rho = |\psi\rangle\langle\psi|$ is a pure qudit state, then $\mathrm{tr}(\rho^2) = 1$, from which it follows that

$$|\vec{c}|^2 = \vec{c} \cdot \vec{c} = D(D-1) \ . \tag{13}$$

We could represent a pure state by a unit vector $\vec{n} = \vec{c}/\sqrt{D(D-1)}$ on the unit sphere in $D^2 - 1$ dimensions, but in contrast to the situation with the Bloch sphere ($D = 2$), most vectors on this unit sphere do not represent a pure state or, indeed, any state at all.

3 Mixtures of maximally mixed and maximally entangled states

In this section we deal with two qudits, labeled A and B. We consider a class of two-qudit states, specifically mixtures of the maximally mixed state, $M_{D^2} = I \otimes I / D^2$, with a maximally entangled state, which we can choose to be

$$|\Psi\rangle = \frac{1}{\sqrt{D}} \sum_{a=1}^{D} |a\rangle \otimes |a\rangle \ . \tag{14}$$

Such mixtures have the form

$$\rho_\epsilon = (1-\epsilon)M_{D^2} + \epsilon |\Psi\rangle\langle\Psi| \ , \tag{15}$$

where $0 \leq \epsilon \leq 1$.

In analogy to Eq. (10), any state ρ of two qudits can be expanded uniquely as

$$\rho = \frac{1}{D^2} c_{\alpha\beta} \lambda_\alpha \otimes \lambda_\beta \ , \tag{16}$$

where the expansion coefficients are given by

$$c_{\alpha\beta} = D^2 \text{tr}(\rho \lambda_\alpha \otimes \lambda_\beta) \ , \tag{17}$$

with $c_{00} = D$ determined by normalization. Using Eq. (17) or Eqs. (7)–(9), we can find the operator expansion for the maximally entangled state (14):

$$|\Psi\rangle\langle\Psi| = \frac{1}{D^2}\left(I \otimes I + D\sum_a \Gamma_a \otimes \Gamma_a + D\sum_{a<b}\left(\Gamma_{ab}^{(+)} \otimes \Gamma_{ab}^{(+)} - \Gamma_{ab}^{(-)} \otimes \Gamma_{ab}^{(-)}\right)\right) \ , \tag{18}$$

from which we can read off the expansion coefficients for the state ρ_ϵ of Eq. (15):

$$c_{0j} = c_{j0} = 0 \ , \tag{19}$$

$$c_{jk} = \begin{cases} 0 \ , & j \neq k, \\ D\epsilon \ , & j = k = 1,\ldots,(D+2)(D-1)/2. \\ -D\epsilon \ , & j = k = D(D+2)/2,\ldots,D^2-1. \end{cases} \tag{20}$$

A state of the two qudits is *separable* if it can be written as an ensemble of product states. In this section we show that the mixed state state (15) is separable if and only if

$$\epsilon \leq \frac{1}{1+D} \ . \tag{21}$$

Our method is to prove the necessity of the condition (21) by considering the restrictions that separability places on the correlation coefficients (20) and then to construct an explicit product ensemble when $\epsilon \leq 1/(1+D)$. Vidal and Tarrach [13] found the separability boundary for a mixture of M_{D^2} with *any* pure state by using the partial transpose condition [8] to show that any state with ϵ outside the boundary is nonseparable and by constructing an explicit product ensemble for states with ϵ within the separability boundary. Horodecki and

Horodecki [12] found the separability boundary for the state (15) using other techniques. The reason for presenting in this section a more limited result than that of Vidal and Tarrach is, first, that our proof of necessity has a nice physical interpretation in terms of the correlation coefficients (20) and, second, that the product ensemble we use is different from the one used by Vidal and Tarrach.

The product pure states for two qudits, $|\psi_A\rangle\langle\psi_A|\otimes|\psi_B\rangle\langle\psi_B|$, constitute an overcomplete operator basis. Thus we can expand any two-qudit density operator in terms of them,

$$\rho = \int d\mathcal{V}_A \, d\mathcal{V}_B \, w(\psi_A, \psi_B) \, |\psi_A\rangle\langle\psi_A|\otimes|\psi_B\rangle\langle\psi_B| \, . \tag{22}$$

Here the integral for each system runs over all of projective Hilbert space, i.e., the space of Hilbert-space rays, and the volume elements $d\mathcal{V}_A$ and $d\mathcal{V}_B$ are the unitarily invariant integration measures on projective Hilbert space. Because of overcompleteness of the pure-state projectors, the expansion function $w(\psi_A, \psi_B)$ is not unique. Notice that the expansion coefficients $c_{\alpha\beta}$ of Eq. (17) can be written as integrals over the expansion function,

$$c_{\alpha\beta} = \int d\mathcal{V}_A \, d\mathcal{V}_B \, w(\psi_A, \psi_B) \, (c_A)_\alpha (c_B)_\beta \, , \tag{23}$$

where $(c_A)_\alpha = D\langle\psi_A|\lambda_\alpha|\psi_A\rangle$ and $(c_B)_\alpha = D\langle\psi_B|\lambda_\alpha|\psi_B\rangle$ are the expansion coefficients for the pure states $|\psi\rangle_A$ and $|\psi\rangle_B$, satisfying $\vec{c}_A \cdot \vec{c}_A = D(D-1) = \vec{c}_B \cdot \vec{c}_B$.

A two-qudit state is separable if and only if there exists an expansion function $w(\psi_A, \psi_B)$ that is everywhere nonnegative. In this case $w(\psi_A, \psi_B)$ can be thought of as a normalized classical probability distribution for the pure states ψ_A and ψ_B, and the integral for $c_{\alpha\beta}$ in Eq. (23) can be interpreted as a classical expectation value of the product of the random variables $(c_A)_\alpha$ and $(c_B)_\beta$, i.e.,

$$c_{\alpha\beta} = E\left[(c_A)_\alpha (c_B)_\beta\right] \, . \tag{24}$$

If the state ρ_ϵ is separable, we have from Eq. (20) that for each value of j,

$$D\epsilon = |c_{jj}| = \left|E\left[(c_A)_j (c_B)_j\right]\right| \leq \frac{1}{2}\left(E[(c_A)_j^2] + E[(c_B)_j^2]\right) \, . \tag{25}$$

Adding over the $D^2 - 1$ value of j gives

$$D(D^2 - 1)\epsilon \leq \frac{1}{2}\left(E[\vec{c}_A \cdot \vec{c}_A] + E[\vec{c}_B \cdot \vec{c}_B]\right) = D(D-1) \, . \tag{26}$$

We conclude that if ρ_ϵ is separable, then $\epsilon \leq 1/(1 + D)$.

To prove the converse, we construct an explicit product ensemble for the state ρ_ϵ with $\epsilon = 1/(1 + D)$. We define a vector $\vec{z} = (z_1, \ldots, z_D)$ whose components z_a take on the values ± 1 and $\pm i$, so that

$$\sum_{z_j} z_j = \sum_{z_j} z_j^2 = 0 \, , \quad \sum_{z_j} |z_j|^2 = 4 \, . \tag{27}$$

Associated with each vector $\vec{z}$ is a pure state

$$|\Phi_{\vec{z}}\rangle = \frac{1}{\sqrt{D}} \sum_{a=1}^{D} z_a|a\rangle \, . \tag{28}$$

There are 4^D vectors and thus that many states $\Phi_{\vec{z}}$, although only 4^{D-1} of these states are distinct in that they differ by more than a global phase. Now we define a product state for the two-qudit system:

$$\rho_{\vec{z}} = |\Phi_{\vec{z}}\rangle\langle\Phi_{\vec{z}}| \otimes |\Phi_{\vec{z}*}\rangle\langle\Phi_{\vec{z}*}| \ . \tag{29}$$

The ensemble consisting of all 4^D of these states, each contributing with the same probability, produces the density operator

$$\frac{1}{4^D}\sum_{\vec{z}}\rho_{\vec{z}} = \frac{1}{4^D D^2}\sum_{a,b,c,d}\left(\sum_{\vec{z}}z_a z_b^* z_c^* z_d\right)|a\rangle\langle b| \otimes |c\rangle\langle d| \ . \tag{30}$$

Since

$$\sum_{\vec{z}}z_a z_b^* z_c^* z_d = 4^D(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} - \delta_{ab}\delta_{cd}\delta_{ac}) \ , \tag{31}$$

it follows that

$$\frac{1}{4^D}\sum_{\vec{z}}\rho_{\vec{z}} = \frac{I \otimes I}{D^2} + \frac{1}{D}|\Psi\rangle\langle\Psi| - \frac{1}{D^2}\sum_{a=1}^{D}|a\rangle\langle a| \otimes |a\rangle\langle a| \ . \tag{32}$$

Multiplying by $D/(D+1)$ and rearranging yields

$$\frac{D}{1+D}\frac{I \otimes I}{D^2} + \frac{1}{1+D}|\Psi\rangle\langle\Psi| = \frac{D}{1+D}\frac{1}{4^D}\sum_{\vec{z}}\rho_{\vec{z}} + \frac{1}{1+D}\frac{1}{D}\sum_{a=1}^{D}|a\rangle\langle a| \otimes |a\rangle\langle a| \ . \tag{33}$$

The left-hand side of Eq. (33) is the state (15) with $\epsilon = 1/(1+D)$, and the right-hand side is an explicit product ensemble for the state. This concludes the proof that ρ_ϵ is separable if and only if $\epsilon \leq 1/(1+D)$.

4 Separability of states near the maximally mixed state

This section deals with N-qudit states of the form

$$\rho_\epsilon = (1-\epsilon)M_{D^N} + \epsilon\rho_1 \ , \tag{34}$$

where $M_{D^N} = I\otimes\cdots\otimes I/D^N$ is the maximally mixed state for N qudits and ρ_1 is any N-qudit density operator. We establish lower and upper bounds on the size of the neighborhood of separable states surrounding the maximally mixed state. In particular, we show, first, that for

$$\epsilon \leq \frac{1}{1+D^{2N-1}} \ , \tag{35}$$

all states of the form (34) are separable and, second, that for

$$\epsilon > \frac{1}{1+D^{N-1}} \ , \tag{36}$$

there are states of the form (34) that are not separable (i.e., they are entangled). These results generalize and extend the work of Braunstein *et al.* for qubits [14] and of Caves and Milburn for qutrits [15].

4.1 Mathematical preliminaries

Before turning to the lower and upper bounds, it is useful to develop some mathematical apparatus that will be used in deriving the bounds.

4.1.1 Superoperator formalism

We begin by reviewing a formalism for handling superoperators, introduced by Caves [16] and used by Schack and Caves [17] to generate product ensembles for separable N-qubit states.

The space of linear operators acting on a D-dimensional complex vector space is a D^2-dimensional complex vector space. In this space we introduce operator "kets" $|A) = A$ and "bras" $(A| = A^\dagger$, distinguished from vector kets and bras by the use of smooth brackets. The natural operator inner product can be written as $(A|B) = \text{tr}(A^\dagger B)$. An orthonormal basis $|a\rangle$ induces an orthonormal operator basis,

$$|c\rangle\langle a| = \tau_{ca} = \tau_\alpha \ , \tag{37}$$

where the Greek index α is an abbreviation for the pair of Roman indices, ca. Not all orthonormal operator bases are of this outer-product form.

The space of superoperators, i.e., linear maps on operators, is a D^4-dimensional complex vector space. Any superoperator $\mathcal{S}$ is specified by its "matrix elements"

$$\mathcal{S}_{ca,db} = \langle c|\,\mathcal{S}(|a\rangle\langle b|)|d\rangle \ , \tag{38}$$

for the superoperator can be written in terms of its matrix elements as

$$\mathcal{S} = \sum_{c,a,d,b} \mathcal{S}_{ca,db}|c\rangle\langle a| \odot |b\rangle\langle d| = \sum_{c,a,d,b} \mathcal{S}_{ca,db}\, \tau_{ca} \odot \tau_{db}^\dagger = \sum_{\alpha,\beta} \mathcal{S}_{\alpha\beta}|\tau_\alpha)(\tau_\beta| \ . \tag{39}$$

The tensor product here is an ordinary operator tensor product, but we use the symbol $\odot$ to distinguish it from a tensor product between objects associated with different systems, which is denoted by $\otimes$. In the final form of Eq. (39), the tensor product is written as an operator outer product, with $\alpha = ca$ and $\beta = db$.

The ordinary action of $\mathcal{S}$ on an operator A, used to generate the matrix elements, is obtained by dropping an operator A into the center of the representation of $\mathcal{S}$, in place of the tensor-product sign,

$$\mathcal{S}(A) = \sum_{\alpha,\beta} \mathcal{S}_{\alpha\beta}\, \tau_\alpha A \tau_\beta^\dagger \ . \tag{40}$$

There is clearly another way that $\mathcal{S}$ can act on A, the left-right action,

$$\mathcal{S}|A) = \sum_{\alpha,\beta} \mathcal{S}_{\alpha\beta}|\tau_\alpha)(\tau_\beta|A) \ , \tag{41}$$

in terms of which the matrix elements are

$$\mathcal{S}_{\alpha\beta} = (\tau_\alpha|\,\mathcal{S}\,|\tau_\beta) = (\tau_{ca}|\,\mathcal{S}\,|\tau_{db}) = \langle c|\,\mathcal{S}(|a\rangle\langle b|)|d\rangle \ . \tag{42}$$

This expression provides the fundamental connection between the two actions of a superoperator. We can define an operation, called *sharp*, that exchanges the ordinary and left-right actions:

$$\mathcal{S}^{\#}(A) = \mathcal{S}|A) \ . \tag{43}$$

Equation (42) implies that

$$\mathcal{S}^{\#}_{ca,db} = \langle c | \mathcal{S}^{\#}(|a\rangle\langle b|)|d\rangle = (\tau_{cd}|\mathcal{S}|\tau_{ab}) = \mathcal{S}_{cd,ab} \tag{44}$$

or, equivalently, that

$$\mathcal{S}^{\#} = \sum_{c,a,d,b} \mathcal{S}_{ca,db} |c\rangle\langle d| \odot |b\rangle\langle a| . \tag{45}$$

With respect to the left-right action, a superoperator works just like an operator. Multiplication of superoperators $\mathcal{R}$ and $\mathcal{S}$ is given by

$$\mathcal{R}\mathcal{S} = \sum_{\alpha,\beta,\gamma} \mathcal{R}_{\alpha\gamma}\mathcal{S}_{\gamma\beta}|\tau_\alpha)(\tau_\beta| , \tag{46}$$

and the adjoint is defined by

$$(A|\mathcal{S}^{\dagger}|B) = (B|\mathcal{S}|A)^* \iff \mathcal{S}^{\dagger} = \sum_{\alpha,\beta} \mathcal{S}^{*}_{\beta\alpha}|\tau_\alpha)(\tau_\beta| . \tag{47}$$

With respect to the ordinary action, superoperator multiplication, denoted as a composition $\mathcal{R} \circ \mathcal{S}$, is given by

$$\mathcal{R} \circ \mathcal{S} = \sum_{\alpha,\beta,\gamma,\delta} \mathcal{R}_{\gamma\delta}\mathcal{S}_{\alpha\beta}\,\tau_\gamma\tau_\alpha \odot \tau_\beta^{\dagger}\tau_\delta^{\dagger} . \tag{48}$$

The adjoint with respect to the ordinary action, denoted by $\mathcal{S}^{\times}$, is defined by

$$\mathrm{tr}([\mathcal{S}^{\times}(B)]^{\dagger}A) = \mathrm{tr}(B^{\dagger}\mathcal{S}(A)) \iff \mathcal{S}^{\times} = \sum_{\alpha,\beta} \mathcal{S}^{*}_{\alpha\beta}\,\tau_\alpha^{\dagger} \odot \tau_\beta . \tag{49}$$

The identity superoperator with respect to the left-right action can be written as

$$\mathbf{I} = \sum_\alpha |\tau_\alpha)(\tau_\alpha| = \sum_{c,a} |c\rangle\langle a| \odot |a\rangle\langle c| . \tag{50}$$

When sharped, $\mathbf{I}$ becomes the identity superoperator with respect to the ordinary action, denoted by $\mathcal{I}$:

$$\mathbf{I}^{\#} = \sum_{c,a} |c\rangle\langle c| \odot |a\rangle\langle a| = I \odot I \equiv \mathcal{I} . \tag{51}$$

The final ingredient we need is the superoperator trace relative to the left-right action, defined by

$$\mathrm{Tr}(\mathcal{S}) = \sum_\alpha (\tau_\alpha|\mathcal{S}|\tau_\alpha) = \sum_{c,a} \langle c|\mathcal{S}(|a\rangle\langle a|)|c\rangle = \mathrm{tr}(\mathcal{S}(I)) . \tag{52}$$

Notice that $\mathbf{I}(I) = DI$ and $\mathcal{I}(I) = I$, which give $\mathrm{Tr}(\mathbf{I}) = D^2$ and $\mathrm{Tr}(\mathcal{I}) = D$.

Now suppose the operators $|N_\alpha)$ constitute a complete or overcomplete operator basis; i.e., let the operator kets $|N_\alpha)$ span the vector space of operators. It follows that the superoperator $\mathcal{G}$ defined by

$$\mathcal{G} = \sum_\alpha |N_\alpha)(N_\alpha| = \mathcal{G}^{\dagger} \tag{53}$$

is invertible with respect to the left-right action. The operators

$$|Q_\alpha) = \mathcal{G}^{-1}|N_\alpha) \tag{54}$$

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form a dual basis, which gives rise to the following expressions for the identity superoperator:

$$\mathbf{I} = \sum_\alpha |Q_\alpha)(N_\alpha| = \sum_\alpha |N_\alpha)(Q_\alpha| \,. \tag{55}$$

An arbitrary operator A can be expanded in terms of the original basis or the dual basis:

$$A = \sum_\alpha |N_\alpha)(Q_\alpha|A) = \sum_\alpha N_\alpha \mathrm{tr}(Q_\alpha^\dagger A) \,, \tag{56}$$

$$A = \sum_\alpha |Q_\alpha)(N_\alpha|A) = \sum_\alpha Q_\alpha \mathrm{tr}(N_\alpha^\dagger A) \,. \tag{57}$$

These expansions are unique if and only if the operators $|N_\alpha)$ are linearly independent. Later in this section we apply expansions of this sort to density operators.

4.1.2 Pure states and their dual basis

The set of all pure-state projectors in a D-dimensional Hilbert space,

$$P_\psi = |\psi\rangle\langle\psi| \,, \tag{58}$$

forms an overcomplete operator basis. To develop operator expansions in terms of the pure-state projectors, we follow the discussion in the preceding subsection and consider the superoperator

$$\mathcal{G} = \int d\mathcal{V} \, |P_\psi)(P_\psi| = \int d\mathcal{V} \, |\psi\rangle\langle\psi| \odot |\psi\rangle\langle\psi| \,, \tag{59}$$

where $d\mathcal{V}$ is the unitarily invariant integration measure on projective Hilbert space.

The only Hilbert-space integrals we need to calculate explicitly are those for which the integrand is a function only of an angle θ defined by $\cos\theta = |\langle e|\psi\rangle|$, where $|e\rangle$ is some particular unit vector (pure state). The angle θ, which runs over the range $0 \le \theta \le \pi/2$, can be thought of as a "polar angle" relative to the "polar axis" defined by $|e\rangle$. For integrals of this sort, a convenient form of the integration measure is [18]

$$d\mathcal{V} = (\sin\theta)^{2D-3} \cos\theta \, d\theta \, d\mathcal{S}_{2D-3} \,, \tag{60}$$

where $d\mathcal{S}_{2D-3}$ is the standard integration measure on a $(2D-3)$-dimensional unit sphere. Thus the total volume of D-dimensional projective Hilbert space is [18]

$$\mathcal{V} = \mathcal{S}_{2D-3} \int_0^{\pi/2} d\theta \, (\sin\theta)^{2D-3} \cos\theta \, d\theta = \frac{\mathcal{S}_{2D-3}}{2(D-1)} = \frac{\pi^{D-1}}{(D-1)!} \,, \tag{61}$$

where $\mathcal{S}_{2D-3} = 2\pi^{D-1}/(D-2)!$ is the volume of a $(2D-3)$-dimensional unit sphere.

To use the expansions (56) and (57), we need the dual basis $|Q_\psi)$, and for that purpose, we need to invert $\mathcal{G}$. Since $\mathcal{G}$ is Hermitian relative to the left-right action, we can invert it by diagonalizing it with respect to the left-right action. Given an orthonormal basis $|a\rangle$, we can write $\mathcal{G}$ as in Eq. (39),

$$\mathcal{G} = \sum_{c,a,d,b} \mathcal{G}_{ca,db}|c\rangle\langle a| \odot |b\rangle\langle d| = \sum_{c,a,d,b} \mathcal{G}_{ca,db}|\tau_{ca})(\tau_{db}| \,, \tag{62}$$

where the matrix elements are given by Eq. (38):

$$\mathcal{G}_{ca,db} = \langle c|\,\mathcal{G}(|a\rangle\langle b|)|d\rangle = \int d\mathcal{V}\,\langle c|\psi\rangle\langle\psi|a\rangle\langle b|\psi\rangle\langle\psi|d\rangle\,. \tag{63}$$

The unitary invariance of the integration measure places stringent constraints on the matrix elements (63). Since the integral in Eq. (63) remains unchanged under a change in the sign of the amplitude $\langle a|\psi\rangle$ corresponding to a particular basis vector $|a\rangle$, the matrix elements vanish except when (i) $a = b \neq c = d$ or $a = c \neq b = d$ or (ii) $a = b = c = d$. Furthermore, unitary invariance implies that for each of these cases, all the matrix elements have the same value. Gathering these conclusions together, we have

$$\mathcal{G}_{ca,db} = \begin{cases} \alpha\,, & a = b \neq c = d \text{ or } a = c \neq b = d, \\ \gamma\,, & a = b = c = d, \\ 0\,, & \text{otherwise}. \end{cases} \tag{64}$$

We get a relation between α and γ by noting that

$$D(D-1)\alpha + D\gamma = \sum_{c,a=1}^{D} \mathcal{G}_{ca,ca} = \mathcal{V}\,, \tag{65}$$

where the second equality follows from doing the sum within the integral in Eq. (63). We need one more relation, which we get by evaluating explicitly the integral for γ:

$$\gamma = \int d\mathcal{V}|\langle a|\psi\rangle|^4 = \mathcal{S}_{2D-3}\int_0^{\pi/2} d\theta\,(\sin\theta)^{2D-3}(\cos\theta)^5 = \frac{2\mathcal{V}}{D(D+1)} \equiv 2K\,. \tag{66}$$

It follows that $\alpha = K$. As a result, we have

$$\mathcal{G} = K\left(2\sum_a |\tau_{aa})(\tau_{aa}| + \sum_{\substack{a,b \\ a\neq b}}\left(|\tau_{ab})(\tau_{ab}| + |\tau_{aa})(\tau_{bb}|\right)\right) = K(\mathbf{I} + \mathcal{I})\,. \tag{67}$$

This result gives us immediately that [18]

$$\int d\mathcal{V}\,|\psi\rangle\langle\psi| = \mathcal{G}(I) = \frac{\mathcal{V}}{D}I\,. \tag{68}$$

The operators λ_α introduced in Sect. 2 constitute a complete, orthonormal operator basis, so we can write $\mathbf{I}$ as

$$\mathbf{I} = \sum_\alpha |\lambda_\alpha)(\lambda_\alpha| = \frac{|I)(I|}{D} + \mathcal{T}\,, \tag{69}$$

where

$$\mathcal{T} = \sum_j |\lambda_j)(\lambda_j| \tag{70}$$

is the superoperator that projects onto the subspace of traceless operators. Plugging Eq. (69) into Eq. (67) gives the diagonal form of $\mathcal{G}$:

$$\mathcal{G} = K\left((D+1)\frac{|I)(I|}{D} + \mathcal{T}\right)\,. \tag{71}$$

Orthonormal eigenoperators of $\mathcal{G}$ are $\lambda_0 = I/\sqrt{D}$, with eigenvalue $K(D+1) = \mathcal{V}/D$ and the traceless operators λ_j, which are degenerate with eigenvalue $K = \mathcal{V}/D(D+1)$.

We are now prepared to write the inverse of $\mathcal{G}$ with respect to the left-right action as

$$\mathcal{G}^{-1} = \frac{1}{K}\left(\frac{1}{D+1}\frac{|I)(I|}{D} + \mathcal{T}\right) = \frac{1}{K}\left(\mathbf{I} - \frac{\mathcal{I}}{D+1}\right) . \tag{72}$$

Thus the dual operators of Eq. (54) are given by

$$|Q_\psi) = \mathcal{G}^{-1}|P_\psi) = \frac{1}{K}\left(|P_\psi) - \frac{|I)}{D+1}\right) = \frac{D}{\mathcal{V}}\big((D+1)P_\psi - I\big) . \tag{73}$$

4.1.3 Alternative diagonalization of $\mathcal{G}$

In this subsection we rederive Eq. (71) using the special properties of the superoperator $\mathcal{G}$. These properties are evident from the integral form of $\mathcal{G}$ in Eq. (59).

- The superoperator $\mathcal{G}$ is Hermitian relative to the left-right action, which implies that it has a complete, orthonormal set of eigenoperators η_α, $\alpha = 1, \ldots, D^2$, with real eigenvalues q_α:

$$\mathcal{G} = \mathcal{G}^\dagger \quad \Longrightarrow \quad \mathcal{G} = \sum_\alpha q_\alpha |\eta_\alpha)(\eta_\alpha| = \sum_\alpha q_\alpha \eta_\alpha \odot \eta_\alpha^\dagger . \tag{74}$$

- The superoperator $\mathcal{G}$ is Hermitian relative to the ordinary action,

$$\mathcal{G} = \mathcal{G}^\times = \sum_\alpha q_\alpha \eta_\alpha^\dagger \odot \eta_\alpha = \sum_\alpha q_\alpha |\eta_\alpha^\dagger)(\eta_\alpha^\dagger| , \tag{75}$$

which implies that if η_α is an eigenoperator of $\mathcal{G}$, then $\eta_\alpha^\dagger$ is also an eigenoperator with the same eigenvalue. This means that we can choose all the eigenoperators to be Hermitian.

- The superoperator $\mathcal{G}$ is unitarily invariant, i.e.,

$$\mathcal{G} = U \odot U^\dagger \circ \mathcal{G} \circ U^\dagger \odot U = \sum_\alpha q_\alpha U\eta_\alpha U^\dagger \odot U\eta_\alpha^\dagger U^\dagger , \tag{76}$$

for any unitary operator U, which implies that if η_α is an eigenoperator of $\mathcal{G}$, then $U\eta_\alpha U^\dagger$ is also an eigenoperator with the same eigenvalue.

The upshot of these three properties is that the eigensubspaces of $\mathcal{G}$ are invariant under Hermitian conjugation and under all unitary transformations. It is not hard to show that the only such operator subspaces are the subspace of traceless operators and its orthocomplement, the one-dimensional subspace spanned by the unit operator. The result is that $\mathcal{G}$ must have the form

$$\mathcal{G} = K\left(\mu\frac{I \odot I}{D} + \mathcal{T}\right) = K\left(\mathbf{I} + \frac{\mu - 1}{D}\mathcal{I}\right) , \tag{77}$$

where K is the eigenvalue of any traceless operator and $K\mu$ is the eigenvalue of $\lambda_0 = I/\sqrt{D}$. Now we use the final property to evaluate μ.

- The superoperator $\mathcal{G}$ is invariant under exchange of the two kinds of action:

$$\mathcal{G} = \mathcal{G}^{\#} = K\left(\mathcal{I} + \frac{\mu - 1}{D}\mathbf{I}\right) . \tag{78}$$

This implies that $\mu = D + 1$, thus bringing $\mathcal{G}$ into the form (67), but with K not yet determined.

We find the value of K by evaluating the superoperator trace, first using Eq. (59),

$$\text{Tr}(\mathcal{G}) = \text{tr}(\mathcal{G}(I)) = \mathcal{V} , \tag{79}$$

and then using Eq. (67),

$$\text{Tr}(\mathcal{G}) = K\big(\text{Tr}(\mathbf{I}) + \text{Tr}(\mathcal{I})\big) = KD(D+1) . \tag{80}$$

This gives $K = \mathcal{V}/D(D+1)$, in agreement with Eq. (66).

4.2   Separability bounds

We turn now to demonstrating the lower and upper bounds, Eqs. (35) and (36), on the size of the neighborhood of separable states surrounding the maximally state.

To establish the lower bound, we use the results of Sect. 4.1 to formulate operator expansions in terms of product pure states. For a single qudit, any density operator can be expanded as

$$\rho = \int d\mathcal{V} \, |P_\psi)(Q_\psi|\rho) = \int d\mathcal{V} \, w_\rho(\psi) P_\psi , \tag{81}$$

where

$$w_\rho(\psi) = \text{tr}(\rho Q_\psi) = \frac{D}{\mathcal{V}}\big((D+1)\langle\psi|\rho|\psi\rangle - 1\big) \tag{82}$$

is a quasi-probability distribution, normalized to unity, but possibly having negative values. The analogous product representation for an N-qudit density operator is

$$\rho = \int d\mathcal{V}_1 \cdots d\mathcal{V}_N \, w_\rho(\psi_1, \ldots, \psi_N) \, P_{\psi_1} \otimes \cdots \otimes P_{\psi_N} , \tag{83}$$

where

$$w_\rho(\psi_1, \ldots, \psi_N) = \text{tr}(\rho Q_{\psi_1} \otimes \cdots \otimes Q_{\psi_N}) . \tag{84}$$

The N-qudit quasi-distribution obeys the bound

$$w_\rho(\psi_1, \ldots, \psi_N) \geq \left(\begin{array}{c}\text{smallest eigenvalue of}\\ Q_{\psi_1} \otimes \cdots \otimes Q_{\psi_N}\end{array}\right) = -\frac{D^{2N-1}}{\mathcal{V}^N} \tag{85}$$

This follows from the fact that Q_ψ has a nondegenerate eigenvalue, $D^2/\mathcal{V}$, and a $(D-1)$-fold degenerate eigenvalue, $-D/\mathcal{V}$. Thus the most negative eigenvalue of the product operator $Q_{\psi_1} \otimes \cdots \otimes Q_{\psi_N}$ is $(-D/\mathcal{V})(D^2/\mathcal{V})^{N-1} = -D^{2N-1}/\mathcal{V}^N$.

We can use the lower bound (85) to place a similar lower bound on the quasi-distribution for the mixed state ρ_ϵ of Eq. (34). Since the quasi-distribution for the maximally mixed state, M_{D^N}, is the uniform distribution $1/\mathcal{V}^N$, we have

$$w_{\rho_\epsilon}(\psi_1, \ldots, \psi_N) = \frac{1-\epsilon}{\mathcal{V}^N} + \epsilon w_{\rho_1} \geq \frac{1 - \epsilon(1 + D^{2N-1})}{\mathcal{V}^N} . \tag{86}$$

We conclude that if $\epsilon \leq 1/(1 + D^{2N-1})$, then w_{ρ_ϵ} is nonnegative and the qudit state ρ_ϵ is separable. This establishes the lower bound (35) on the size of the neighborhood of separable states surrounding the maximally mixed state.

The upper bound (36) on the size of the separable neighborhood can be established with the help of an exact separability condition for a particular N-qubit state, obtained by Dür, Cirac, and Tarrach [19] and also by Pittenger and Rubin [20]. We consider the N-qudit state,

$$\rho_\epsilon = (1 - \epsilon)M_{D^N} + \epsilon|\Psi_{\text{cat}}\rangle\langle\Psi_{\text{cat}}| , \tag{87}$$

where

$$|\Psi_{\text{cat}}\rangle = \frac{1}{\sqrt{D}} \sum_{a=1}^{D} |a\rangle \otimes \cdots \otimes |a\rangle , \tag{88}$$

is an N-qudit "cat state." We call the mixed state (87) an ϵ-cat state.

Now project each qudit onto the two-dimensional (qubit) subspace spanned by $|1\rangle$ and $|2\rangle$. The local projection operator on each qudit is $\Pi = |1\rangle\langle 1| + |2\rangle\langle 2|$, and the normalized N-qubit state after projection is

$$\rho'_\epsilon = \frac{\Pi^{\otimes N} \rho_\epsilon \Pi^{\otimes N}}{\text{tr}(\Pi^{\otimes N} \rho_\epsilon)} = (1 - \epsilon')M_{2^N} + \epsilon'|\Phi_{\text{cat}}\rangle\langle\Phi_{\text{cat}}| , \tag{89}$$

where

$$|\Phi_{\text{cat}}\rangle \equiv \frac{1}{\sqrt{2}}\Big(|1\rangle \otimes \ldots \otimes |1\rangle + |2\rangle \otimes \ldots \otimes |2\rangle\Big) \tag{90}$$

is the cat state for N qubits and

$$\epsilon' = \frac{2\epsilon/D}{(2/D)^N(1 - \epsilon) + 2\epsilon/D} . \tag{91}$$

Dür, Cirac, and Tarrach [19] and also Pittenger and Rubin [20] have shown that the N-qubit ϵ-cat state (89) is nonseparable (entangled) if and only if $\epsilon' > 1/(1 + 2^{N-1})$, a condition equivalent to $\epsilon > 1/(1 + D^{N-1})$. Since local projections on each qudit cannot create entanglement, we can conclude that the N-qudit ϵ-cat state (87) is nonseparable under the same condition. This establishes the upper bound (36) on the size of the separable neighborhood around the maximally mixed state.

Pittenger and Rubin [21] have recently extended the result of Dür, Cirac, and Tarrach [19] for the N-qubit ϵ-cat state. They have shown directly that the N-qudit ϵ-cat state (87) is nonseparable if $\epsilon > 1/(1 + D^{N-1})$, and they have also shown that the same condition is a necessary and sufficient condition for entanglement when D is prime. Their argument is akin to the correlation-coefficient argument we give in Sect. 3.

References

[1] *Introduction to Quantum Computation and Information*, edited by H.-K. Lo, S. Popescu, and T. Spiller (World Scientific, Singapore, 1998).

[2] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).

[3] P. W. Shor, Phys. Rev. A **52**, R2493 (1995).

[4] A. M. Steane, Phys. Rev. Lett. **77**, 793 (1996).

[5] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).

[6] J. S. Bell, Physics (N.Y.) **1**, 195 (1964).

[7] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).

[8] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).

[9] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).

[10] P. Horodecki, Phys. Lett. A **232**, 333 (1997).

[11] M. Lewenstein, J. I. Cirac, and S. Karnas, unpublished, `quant-ph/9903012`.

[12] M. Horodecki and P. Horodecki, Phys. Rev. A **59**, 4206 (1999).

[13] G. Vidal and R. Tarrach, Phys. Rev. A **59**, 141 (1999).

[14] S. L. Braunstein, C. M. Caves, R. Jozsa, N. Linden, S. Popescu, and R. Schack, Phys. Rev. Lett. **83**, 1054 (1999).

[15] C. M. Caves and G. J. Milburn, Opt. Commun., to be published, `quant-ph/9910001`.

[16] C. M. Caves, J. Superconductivity **12**, 707 (1999).

[17] R. Schack and C. M. Caves, J. Mod. Opt., to be published, `quant-ph/9904109`.

[18] R. Schack, G. M. D'Ariano, and C. M. Caves, Phys. Rev. E **50**, 972 (1994).

[19] W. Dür, J. I. Cirac, and R. Tarrach, Phys. Rev. Lett. **83**, 3562 (1999); see also W. Dür and J. I. Cirac, unpublished, `quant-ph/9911044`.

[20] A. O. Pittenger and M. H. Rubin, unpublished, `quant-ph/9912116`.

[21] A. O. Pittenger and M. H. Rubin, unpublished, `quant-ph/0001014`.