Continuous-variable Einstein-Podolsky-Rosen paradox with traveling-wave second-harmonic generation

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(Received 14 March 2004; published 22 September 2004)

The Einstein-Podolsky-Rosen paradox and quantum entanglement are at the heart of quantum mechanics. Here we show that single-pass traveling-wave second-harmonic generation can be used to demonstrate both entanglement and the paradox with continuous variables that are analogous to the position and momentum of the original proposal.

DOI: 10.1103/PhysRevA.70.035801 PACS number(s): 42.50.Xa, 42.50.Lc, 42.50.Dv, 03.65.Ud

I. INTRODUCTION

The early days of quantum mechanics saw many debates about the applicability of classical concepts, such as position and momentum. Two of the main protagonists were Einstein and Bohr, with Einstein, Podolsky, and Rosen (EPR) publishing a famous paper in 1935 [1], setting out what became known as the Einstein-Podolsky-Rosen paradox, and claiming that quantum mechanics was incomplete. Bohr replied with a paper arguing that EPR did not establish this incompleteness, but rather showed the inapplicability of classical descriptions in the quantum domain [2]. Although Bell suggested a set of inequalities that could be violated by quantum mechanics, but not by local hidden variable theories [3], a direct and feasible demonstration of the EPR paradox with continuous variables was first suggested by Reid [4]. The proposal was for an optical demonstration of the paradox via quadrature phase amplitudes, using nondegenerate parametric amplification (also known as the OPA), and was closely related to the original version, which considered position and momentum. The essential step in the EPR argument is to introduce correlated (entangled) states of at least two particles which persist even when the particles become spatially separated. According to EPR, depending on which property of one group of particles that we choose to measure, we can predict with some certainty the values of physical quantities of the other group of particles. If these properties are represented by noncommuting operators (such as position and momentum, or quadrature amplitudes), we may seemingly violate the Heisenberg uncertainty principle. The EPR conclusion was therefore that the description of physical reality given by quantum mechanics is not complete.

Quantum optics was one of the first areas of physics which allowed for simple investigations of some of these fundamental mysteries and paradoxes of quantum mechanics [5]. Among the simplest possible quantum optical systems which can exhibit nonclassical behaviour are traveling-wave second-harmonic generation (SHG) and parametric downconversion. The process of traveling-wave downconversion has been of special interest because it allows for many experiments concerned with the fundamentals of quantum mechanics, among these violations of Bell's inequalities [6–8] and the closely related topic of preparation of Einstein-Podolsky-Rosen states [4,9]. Intracavity second-harmonic generation with transverse degrees of freedom has also been suggested as a source of EPR correlations [10]. What we will demonstrate in this article is that the experimentally simple system of traveling-wave SHG is also a good candidate for a demonstration of the EPR paradox.

II. THE SYSTEM AND EQUATIONS OF MOTION

The system of interest couples two electromagnetic fields via a nonlinear medium (normally a crystal), with a second-order susceptibility represented by $\chi^{(2)}$. The medium is pumped with a continuous wave laser at a frequency ω_1, which interacts to produce a field at frequency $\omega_2(=2\omega_1)$. We follow the approach of Huttner *et al.* [11], quantizing the two interacting fields in terms of the photon fluxes rather than in terms of energy densities. As stated in Ref. [11], this approach avoids problems which could arise, especially with the quantization volume, if we were to work with the normal Hamiltonian approach. With the appropriate momentum-space operators, we use the well-known mapping onto stochastic differential equations in the positive-*P* representation [12] to calculate the development of the fields as they traverse the medium. We consider here the case of one dimensional propagation, which is valid for the case of colinear pumping.

In this approach, the operator

$$\hat{N}(z_0,\omega_m) \equiv \hat{a}^\dagger(z_0,\omega_m)\hat{a}(z_0,\omega_m), \tag{1}$$

for example, is the number operator for photons at frequency ω_m which pass through a plane at $z=z_0$ during a chosen time interval. The bosonic operators $\hat{a}^\dagger(z,\omega_m)$ and $\hat{a}(z,\omega_m)$ obey spatial commutation relations (see also Caves and Crouch [13]), $[\hat{a}(z,\omega_i),\hat{a}^\dagger(z',\omega_j)]=\delta_{ij}\delta(z-z')$. This approach allows us to define the linear free-space momentum operator,

$$\hat{G}_l(z) = \sum_m \hbar k_m \hat{a}^\dagger(z,\omega_m)\hat{a}(z,\omega_m), \tag{2}$$

where $k_m=\omega_m/c$ is the free space wave vector. The above operator represents the number of photons times their individual momentum, thus giving the total momentum of the field passing through the plane during the period of time considered.

Inside the medium, with second order susceptibility $\chi^{(2)}$, and assuming phase-matching at the central frequencies of the two fields, the nonlinear momentum operator is

$$\hat{G}_{nl}(z) = i\hbar\kappa[\hat{a}^2(z,\omega_1)\hat{b}^\dagger(z,\omega_2) - \hat{a}^{\dagger 2}(z,\omega_1)\hat{b}(z,\omega_2)], \quad (3)$$

where

$$\kappa = \frac{\chi^{(2)}}{2\epsilon_0 c}\left[\frac{\omega_1\omega_2}{n(\omega_1)n(\omega_2)}\right]^{1/2}, \quad (4)$$

with the $n(\omega_j)$ being the refractive indices at each frequency and ϵ_0 and c having their usual meanings.

As shown by Shen [14], we can write an equation of motion for the density matrix of the system,

$$i\hbar\frac{\partial\rho(z)}{\partial z} = [\rho(z),\hat{G}_{nl}(z)], \quad (5)$$

which allows for the calculation of steady-state propagation, exactly as required for continuous pumping. Physically, the density matrix, $\rho(z)$, describes an ensemble of steady-state systems which has all the statistical properties of the fields at point z. Equation (5) provides a full description of the interacting fields, but is extremely difficult to solve directly. A commonly used method is to linearize the equations around the semiclassical mean values of the operators, and solve the resulting c-number equations. This has previously been shown to have limited validity for this system, giving erroneous results after a certain interaction length [15,16], but does allow for analytical expressions, which we can compare with the full solutions.

We proceed by mapping the master equation onto a set of stochastic differential equations via the Fokker-Planck equation for the positive-P pseudoprobability distribution of the system. Following the usual methods [17], and making the correspondences $(\hat{a},\hat{a}^\dagger,\hat{b},\hat{b}^\dagger)\leftrightarrow(\alpha,\alpha^+,\beta,\beta^+)$, we find

$$\frac{d\alpha}{dz} = \kappa\alpha^+\beta + \sqrt{\kappa\beta}\;\eta_1(z),$$

$$\frac{d\alpha^+}{dz} = \kappa\alpha\beta^+ + \sqrt{\kappa\beta^+}\;\eta_2(z),$$

$$\frac{d\beta}{dz} = -\frac{\kappa}{2}\alpha^2,$$

$$\frac{d\beta^+}{dz} = -\frac{\kappa}{2}\alpha^{+2}, \quad (6)$$

where the η_i are real Gaussian noise terms with the correlations $\overline{\eta_i(z)\eta_j(z')} = \delta_{ij}\delta(z-z')$. As always with the positive-P, the pairs of field variables (α and α^+ for example) are not complex conjugate except in the mean of a large number of integrated trajectories.

III. QUANTUM CORRELATIONS

A demonstration of the EPR paradox using a nondegenerate OPA has been outlined by Reid [4], and an entanglement criterion for optical quadratures has been outlined by Dechoum *et al.* [18] which follows from inseparabilty criteria developed by Duan *et al.* [19]. We will briefly outline these criteria here and then apply them to our system. In this approach the quadrature operators $\hat{X}_{a,b}$ and $\hat{Y}_{a,b}$, where $\hat{X}_a = \hat{a}+\hat{a}^\dagger$ and $\hat{Y}_a = -i(\hat{a}-\hat{a}^\dagger)$, take the place of the position and momentum considered in the original treatment [1]. Note that these quadratures have the same mathematical properties as the canonical position and momentum operators for the harmonic oscillator, but correspond physically to the real and imaginary parts of the electromagnetic field, not its position and momentum. As shown by Reid, we can make linear estimates of the quadrature variances, which are minimized to give the inferred variances,

$$V^{inf}(\hat{X}_a) = V(\hat{X}_a) - \frac{[V(\hat{X}_a,\hat{X}_b)]^2}{V(\hat{X}_b)},$$

$$V^{inf}(\hat{Y}_a) = V(\hat{Y}_a) - \frac{[V(\hat{Y}_a,\hat{Y}_b)]^2}{V(\hat{Y}_b)},$$

$$V^{inf}(\hat{X}_b) = V(\hat{X}_b) - \frac{[V(\hat{X}_a,\hat{X}_b)]^2}{V(\hat{X}_a)},$$

$$V^{inf}(\hat{Y}_b) = V(\hat{Y}_b) - \frac{[V(\hat{Y}_a,\hat{Y}_b)]^2}{V(\hat{Y}_a)}, \quad (7)$$

where $V(A,B) = \langle AB\rangle - \langle A\rangle\langle B\rangle$. As the $\hat{X}$ and $\hat{Y}$ operators for the same field do not commute, the products of the variances obey a Heisenberg uncertainty relation, with $V(\hat{X})V(\hat{Y}) \geqslant 1$. Hence we find a demonstration of the EPR paradox whenever

$$V^{inf}(\hat{X})V^{inf}(\hat{Y}) \leqslant 1. \quad (8)$$

Entanglement between the modes can be shown using the criterion of Duan *et al.* [19], based on the inseparability of the density matrix. Defining the combined quadratures $\hat{X}_- = \hat{X}_a - \hat{X}_b$ and $\hat{Y}_+ = \hat{Y}_a + \hat{Y}_b$, entanglement is guaranteed provided that

$$V(\hat{X}_-) + V(\hat{Y}_+) < 4. \quad (9)$$

With the quadrature definitions used here, one of these variances individually being less than two signifies two-mode squeezing of the field. For the $\hat{X}_-$ quadrature, this was previously demonstrated by Olsen and Horowicz, using a normalization such that a value of less than one signified two-mode squeezing [20].

IV. RESULTS

We begin by giving linearized results for the criteria defined in the previous section. Assuming a real input coherent state, $\alpha(0)$, with $\beta(0)=0$, we follow the approach used by Ou [21], and also in Ref. [20]. We define a dimensionless interaction length, $\xi=|\alpha(0)|\kappa z/\sqrt{2}$, and fluctuation operators such that $\delta\hat{X}=\hat{X}-\langle\hat{X}\rangle$ and $\delta\hat{Y}=\hat{Y}-\langle\hat{Y}\rangle$, so that, for example, in the linearized approximation, $V(\hat{X})=\delta^2\hat{X}$. The solutions for these fluctuation operators are known [21],

$$\delta\hat{X}_a(\xi) = (1-\xi\tanh\xi)\delta\hat{X}_a(0)\operatorname{sech}\xi + \sqrt{2}\delta\hat{X}_b(0)\tanh\xi\operatorname{sech}\xi,$$

$$\delta\hat{X}_b(\xi) = -\frac{1}{\sqrt{2}}(\tanh\xi+\xi\operatorname{sech}^2\xi)\delta\hat{X}_a(0) + \delta\hat{X}_b(0)\operatorname{sech}^2\xi,$$

$$\delta\hat{Y}_a(\xi) = \delta\hat{Y}_a(0)\operatorname{sech}\xi + \frac{1}{\sqrt{2}}(\sinh\xi+\xi\operatorname{sech}\xi)\delta\hat{Y}_b(0),$$

$$\delta\hat{Y}_b(\xi) = -\sqrt{2}\delta\hat{Y}_a(0)\tanh\xi + (1-\xi\tanh\xi)\delta\hat{Y}_b(0), \tag{10}$$

along with the input correlations,

$$\langle\delta\hat{X}_i(0)\delta\hat{X}_j(0)\rangle = \langle\delta\hat{Y}_i(0)\delta\hat{Y}_j(0)\rangle = \delta_{ij},$$

$$\langle\delta\hat{X}_i(0)\delta\hat{Y}_j(0)\rangle = \langle\delta\hat{Y}_i(0)\delta\hat{X}_j(0)\rangle = 0,$$

$$\langle\delta\hat{X}_i(0)\delta\hat{Y}_i(0)\rangle + \langle\delta\hat{Y}_i(0)\delta\hat{X}_i(0)\rangle = 0, \tag{11}$$

which provides all the information needed to calculate the desired correlations.

To express the inferred variances in their linearized form, we use

$$V_{lin}(\hat{X}_a) = (1-\xi\tanh\xi)^2\operatorname{sech}^2\xi + 2\tanh^2\xi\operatorname{sech}^2\xi,$$

$$V_{lin}(\hat{X}_b) = \frac{1}{2}(\tanh\xi+\xi\operatorname{sech}^2\xi)^2 + \operatorname{sech}^4\xi,$$

$$V_{lin}(\hat{Y}_a) = \operatorname{sech}^2\xi + \frac{1}{2}(\sinh\xi+\xi\operatorname{sech}\xi)^2,$$

$$V_{lin}(\hat{Y}_b) = 2\tanh^2\xi + (1-\xi\tanh\xi)^2,$$

$$V_{lin}(\hat{X}_a,\hat{X}_b) = -\frac{1}{\sqrt{2}}(1-\xi\tanh\xi)(\tanh\xi+\xi\operatorname{sech}^2\xi)\operatorname{sech}\xi + \sqrt{2}\tanh\xi\operatorname{sech}^2\xi,$$

$$V_{lin}(\hat{Y}_a,\hat{Y}_b) = -\sqrt{2}\tanh\xi\operatorname{sech}\xi + \frac{1}{\sqrt{2}}(1-\xi\tanh\xi)(\sinh\xi + \xi\operatorname{sech}\xi). \tag{12}$$

Noting that $V(\hat{X}_-)=V(\hat{X}_a)+V(\hat{X}_b)-2V(\hat{X}_a,\hat{X}_b)$, and that $V(\hat{Y}_+)=V(\hat{Y}_a)+V(\hat{Y}_b)+2V(\hat{Y}_a,\hat{Y}_b)$, we have all that we need to calculate the linearized expressions necessary to demonstrate entanglement and the EPR paradox. However, these expressions get rather bulky and are not very enlightening, so we will plot them for comparison with the results of stochastic integration of the full positive-*P* equations.

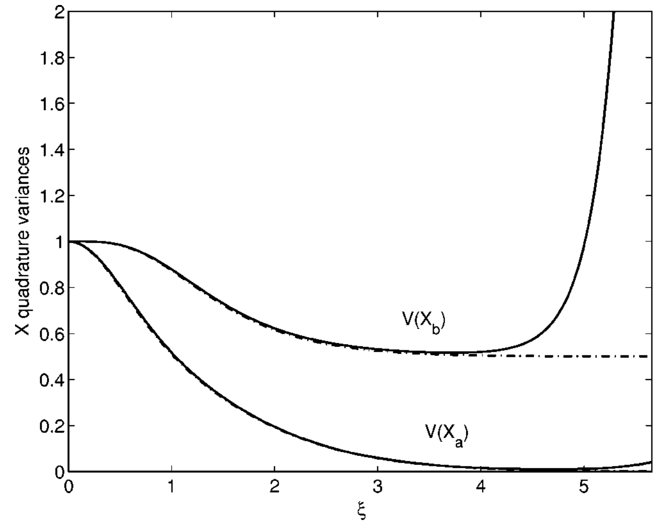

FIG. 1. Variances in the $\hat{X}_a$ and $\hat{X}_b$ quadratures. The solid lines are the stochastic predictions and the dashed lines are the linearized solutions. All quantities plotted in this and subsequent graphs are dimensionless.

The positive-*P* representation equations (6) were integrated numerically using a three-step predictor corrector method, for parameters $\alpha(0)=10^3$, $\beta(0)=0$ and $\kappa=0.01$, with the results being averaged over 1.029×10^7 trajectories. In Fig. 1, we show the results for $V(\hat{X}_a)$ and $V(\hat{X}_b)$, demonstrating that the linearized approximation loses its validity after a certain interaction length. Note that results for these quantities have been shown previously, in Refs. [15,16,20]. The variances in the two $\hat{Y}$ quadratures always exhibit excess noise for $\xi>0$.

In Fig. 2 we compare the positive-*P* representation predictions for the inferred variances with the linearized predictions. Again the two methods agree up to a certain interaction length, but the linearized results predict a monotonically increasing inferred violation of the uncertainty principle, while the stochastic prediction shows that the violation eventually disappears. As with the increase of $V(\hat{X}_b)$ shown in Fig. 1, this begins to happen at the point where downconversion of the harmonic field begins to be important. As this is initially a spontaneous process, the nonclassical correlations are degraded, with the EPR criteria being more sensitive than the individual variances.

Finally, in Fig. 3, we examine entanglement between the modes, using the criterion of Duan *et al.* In this case the stochastic and linearized predictions are identical over the interaction range shown. We find that there is a range of interaction strength over which $V(\hat{X}_-)+V(\hat{Y}_+)<4$, thus meeting the criterion. None of the other quadrature combinations investigated gave values of less than 4. We note here that the result for $V(\hat{X}_-)$, has been given previously, in Ref. [20], but was not investigated in the context of entanglement. What is

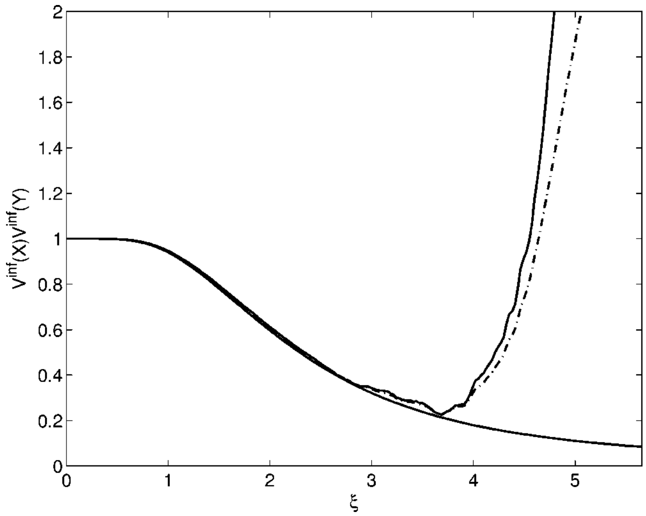

FIG. 2. Products of the inferred variances, giving a clear demonstration of the EPR paradox. The irregular solid line is $V^{inf}(\hat{X}_a)V^{inf}(\hat{Y}_a)$, the dash-dotted line is $V^{inf}(\hat{X}_b)V^{inf}(\hat{Y}_b)$, and the smooth solid line shows the linearized predictions for these two products. The irregularity in the stochastic predictions is the result of averaging over a finite number of stochastic trajectories.

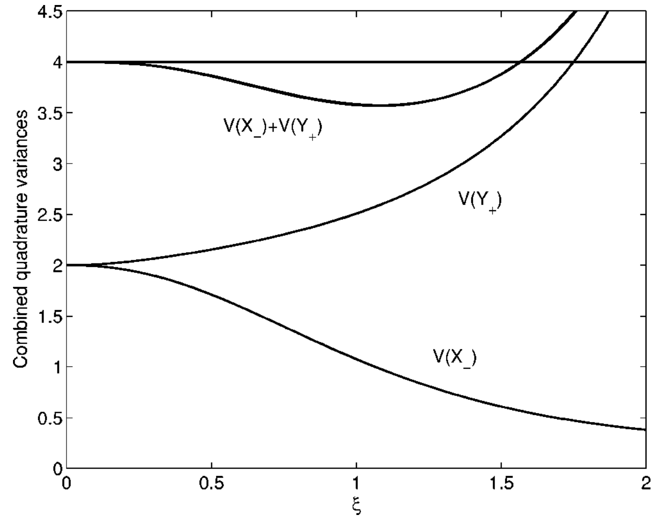

FIG. 3. Variances of the combined quadratures, $\hat{X}_-$ and $\hat{Y}_+$, and the sum of these. The line at a value of 4 is a guide to the eye.

unusual here by comparison with the OPA is that only $V(\hat{X}_-)$ demonstrates squeezing, with $V(\hat{Y}_+)$ always being antisqueezed. In the OPA, both these variances are predicted to be equal and both go to zero in the undepleted pump approximation. Entanglement is found in the present case only because the squeezing in $\hat{X}_-$ is stronger than the antisqueezing in $\hat{Y}_+$. What is also interesting here is that, comparing Fig. 2 and Fig. 3, we see that the EPR paradox is predicted over a greater range of interaction strength than is entanglement.

V. CONCLUSION

In conclusion, we have performed a fully quantum analysis of continuous-wave single-pass second-harmonic generation and compared it with a semiclassical linearized analysis. These analyses show that this simple system may be a good candidate for experimental demonstrations of both quantum entanglement and the Einstein-Podolsky-Rosen paradox with continuous variables. Suggestions as to how these correlations may be measured are given in the literature, for example, in Ref. [4]. We believe that a demonstration should be possible with lasers and nonlinear crystals which are presently available.

ACKNOWLEDGMENTS

This research was supported by the New Zealand Foundation for Research, Science and Technology (Grant. No. UFRJ0001) and the Australian Research Council.

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