Entanglement properties of degenerate four-wave mixing of matter waves in a periodic potential

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In a recent experiment, Campbell *et al.* [Phys. Rev. Lett. **96**, 020406 (2006)] observed degenerate four-wave mixing of matter waves in a one-dimensional optical lattice. We analyze the essential quantum features of the experiment to show that entanglement is created between the quadratures of the two scattered atomic clouds and is a true many-body (rather than two-body) phenomenon. We demonstrate that well-known bipartite entanglement inequalities are significantly violated and that this is robust to a moderate level of coherent seeding. The system is thus a promising candidate for generating spatially separated macroscopically entangled atomic samples.

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I. INTRODUCTION

A recent development in the field of quantum atom optics has been the proposal by Hilligsøe and Mølmer of degenerate four-wave mixing of a Bose-Einstein condensate (BEC) in a periodic potential [1]. The usual quadratic matter-wave dispersion relation in free space ordinarily prevents collisional processes in a single condensate generating new momentum components, although four-wave mixing has been analyzed and demonstrated via collisions of distinct condensates generated using Bragg scattering [2,3]. However, the dispersion relation of a periodic potential can allow phase-matched two-body collision processes within a moving single condensate that conserve both quasimomentum and energy, resulting in the generation of condensates with new momenta. The proposal of Hilligsøe and Mølmer [1] was recently implemented experimentally by Campbell *et al.* [4] with a BEC loaded into a one-dimensional optical lattice. When the phase-matching conditions for energy and quasimomentum were satisfied, both spontaneous and stimulated scattering were observed, with an initial state with one quasimomentum being scattered into two distinct quasimomentum states. Recent experimental work by Gemelke *et al.* [5] has also investigated phase-matched scattering processes of matter waves in a driven optical lattice.

Hilligsøe and Mølmer [1] used the mean-field Gross-Pitaevskii equation (GPE) to analyze a one-dimensional BEC moving in an optical lattice. Although multimode, the GPE cannot describe spontaneous scattering processes and the calculations in Ref. [1] required the initial state to be seeded numerically by hand. In addition, a mean-field approach cannot be used to determine the quantum correlations that are necessary to show entanglement between the scattered momentum states. In this paper, we complement the approach of Ref. [1] by performing fully quantum analyses of the dynamics resulting from a simple three-mode description of the degenerate four-wave-mixing process. While idealized, our analysis allows us to demonstrate continuous variable entanglement without the complications arising from a full multimode, multidimensional analysis.

In terms of the entanglement produced by this process, we note that Campbell *et al.* [4] suggest that "parametric amplification could also be an efficient means of producing pairs of momentum entangled atoms for quantum information applications." Although individual pairs of atoms become correlated in the scattering process, the fact that they are bosonic and scatter into already occupied modes means that the question of which pair that a particular atom belongs to is meaningless. It is also difficult to know how individual pairs (if detected) could be used for quantum information processes as this requires a second degree of freedom such as spin [6,7] that is not present in this experiment. This means that as soon as more than the first two atoms are scattered, pair correlations cannot be measured, even in principle. However, the coherent nature of the scattering allows for the buildup of many-body entanglement between the field quadratures, which, while presently difficult to measure experimentally, is robust to losses and seeding. Although other methods have been proposed and demonstrated for the generation of pair-correlated and entangled atoms from BEC [6–11], the method of Refs. [1,4] appeals because of its relative simplicity. The results discussed below suggest that degenerate four-wave mixing could be an efficient way to generate entangled atomic samples.

II. FORMALISM

A. Description of model

We consider a condensate adiabatically loaded into a one dimensional periodic potential in a single Bloch state with quasi-momentum $\hbar k_0$. For particular combinations of lattice depth and k_0, there exists a phase-matched process that conserves both energy and quasi-momentum such that

$$2k_0 = k_1 + k_2, \quad 2\epsilon(k_0) = \epsilon(k_1) + \epsilon(k_2), \tag{1}$$

where the generated Bloch modes are k_1 and k_2, and $\epsilon(k_i)$ is the energy of mode k_i. Expanding the full Hamiltonian in terms of Bloch states and using a rotating wave approximation, the interaction picture Hamiltonian is

$$\mathcal{H}_{int} = i\hbar\chi[\hat{a}_0^2\hat{a}_1^\dagger\hat{a}_2^\dagger - \hat{a}_0^{\dagger 2}\hat{a}_1\hat{a}_2], \tag{2}$$

where $\hat{a}_i$ is the annihilation operator for quasimomentum mode k_i, and we have made the transformation $\hat{a}_0 \rightarrow \hat{a}_0 e^{i\pi/4}$.

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The strength of the nonlinear interaction is represented by χ, and is given by

$$\chi = \frac{U_0}{\hbar A_\perp} \int dx u_0(x)^2 u_1(x) u_2(x), \tag{3}$$

where $U_0 = 4\pi\hbar^2 a/m$, with a the s-wave scattering length, $A_\perp$ is the cross-sectional area of the system, and $u_i(x)$ is the amplitude of the Bloch state of mode k_i.

We have made several approximations here. The first is that the Bloch states are a good approximation to the eigenstates of the system. This will be true as long as the effective interaction strength χ is sufficiently small. Using mean-field Bloch states appropriate to the effective potential of the lattice plus initial density will extend the regime of validity for short times while the overall density is unchanged, but will alter the phase matching conditions due to the energy shifts of the Bloch modes. The three-mode reduction is also only appropriate for short times before scattering into other modes becomes significant, while the dimensional reduction is appropriate for times short enough that there are no appreciable dynamics in the perpendicular dimensions.

B. Measures of entanglement

Quantum entanglement is a rigorously defined concept which requires that the system density matrix not be separable and is usually demonstrated by the violation of an appropriate inequality. The fact that equal numbers of atoms are scattered from k_0 into k_1 and k_2 (without seeding) is not sufficient to demonstrate entanglement, as this in itself only involves the diagonal elements of the density matrix and can be explained in a completely classical manner. We note here that entanglement was initially explained by Schrödinger in terms of superpositions, which have no classical analogue, and is not compatible with classical ideas such as local realism [12].

To demonstrate entanglement we calculate both the Duan criteria [13] (see also Simon [14]) and a set of Einstein-Podolsky-Rosen (EPR) criteria developed by Reid [15], both of which establish the presence of continuous variable bipartite entanglement. The criteria developed by Duan and Simon are both necessary and sufficient to demonstrate entanglement for Gaussian variables, whereas the EPR criteria are merely sufficient. These many-body continuous-variable criteria are more appropriate to the present case than any consideration of entanglement between the individual atoms of each scattered pair. This type of many-body entanglement is also more robust to dissipation [11] as the loss of a small number of atoms does not markedly affect the values of the quadratures.

For quadrature entanglement, operational criteria have been outlined by Dechoum *et al.* [16], which follow from inequalities developed by Duan *et al.* [13] based on the inseparability of the system density matrix. We briefly outline these criteria here, using the field quadrature operators

$$\hat{X}_i = \hat{a}_i + \hat{a}_i^\dagger, \quad \hat{Y}_i = -i(\hat{a}_i - \hat{a}_i^\dagger). \tag{4}$$

To demonstrate entanglement between the modes, we define the combined quadratures $\hat{X}_\pm = \hat{X}_1 \pm \hat{X}_2$ and $\hat{Y}_\pm = \hat{Y}_1 \pm \hat{Y}_2$. Following the treatment of Ref. [16], entanglement is guaranteed provided that

$$V(\hat{X}_\pm) + V(\hat{Y}_\mp) < 4, \tag{5}$$

where $V(X_i)$ is the variance of X_i. We note here that we are using a specific form of the Duan inequality which is most appropriate for our system in the spontaneous regime, due to the fact that the particles are scattered symmetrically into each mode. In the seeded case, as we are only considering small amounts of seeding, the inequality used is sufficient to show entanglement.

To examine the utility of the system for the production of states which exhibit the EPR paradox [17], we use the method developed by Reid [15]. We assume that a measurement of (for example) the $\hat{X}_1$ quadrature will allow us to infer with some error the value of the $\hat{X}_2$ quadrature, and similarly for the $\hat{Y}_i$ quadratures. By minimizing the rms error in these estimates we find the inferred variances

$$V^{\mathrm{inf}}(\hat{X}_1) = V(\hat{X}_1) - \frac{[V(\hat{X}_1, \hat{X}_2)]^2}{V(\hat{X}_2)},$$

$$V^{\mathrm{inf}}(\hat{Y}_1) = V(\hat{Y}_1) - \frac{[V(\hat{Y}_1, \hat{Y}_2)]^2}{V(\hat{Y}_2)}, \tag{6}$$

with those for the k_2 momentum mode being found by swapping the indices 1 and 2. As the $\hat{X}_i$ and $\hat{Y}_i$ operators do not commute, the products of the actual variances obey a Heisenberg uncertainty relation, with $V(\hat{X}_i)V(\hat{Y}_i) \geq 1$. Hence we find a demonstration of the EPR paradox whenever

$$V^{\mathrm{inf}}(\hat{X}_i) V^{\mathrm{inf}}(\hat{Y}_i) < 1. \tag{7}$$

We therefore see that when the product of these inferred variances falls below one, bipartite entanglement is present between the different momentum modes.

III. RESULTS

A. Analytic approximation

In the limit that only a small number of atoms are scattered from the initial condensate k_0, we can make use of the parametric or undepleted-pump approximation of quantum optics. This approximation has also been utilized in the case of coupled atomic and molecular BEC [18]. Setting $\kappa = \chi\langle \hat{a}_0^2 \rangle$ in the Hamiltonian (2) with κ real results in the following Heisenberg equations of motion

$$\frac{d\hat{a}_1}{dt} = \kappa \hat{a}_2^\dagger, \quad \frac{d\hat{a}_2}{dt} = \kappa \hat{a}_1^\dagger, \tag{8}$$

along with their Hermitian conjugates. The solutions to these equations are well known from quantum optics [19], and provide all the operator moments needed to calculate the entanglement criteria within this approximation. With the two modes k_1 and k_2 initially unpopulated, we find

$$V(\hat{X}_-) + V(\hat{Y}_+) = 4(\cosh 2\kappa t - 2\cosh \kappa t \sinh \kappa t),$$

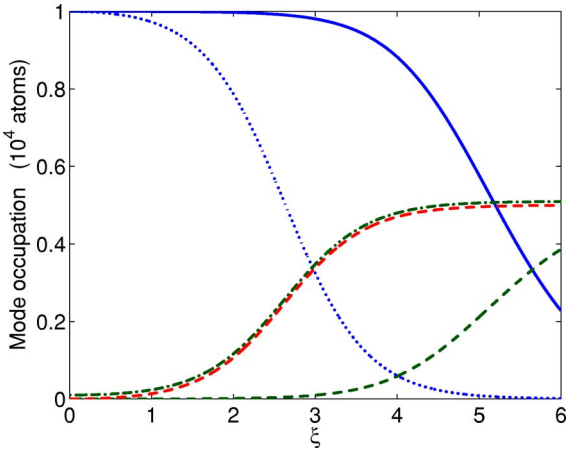

FIG. 1. (Color online) The mode occupations as a function of time. The solid line and the lower dashed line are the averages of 4.34×10^6 trajectories of the positive-P representation equations for N_0, N_1, and N_2 in the spontaneous case. Note that $N_1=N_2$ and that the Wigner results are indistinguishable. The dotted line and the upper dashed lines are the Wigner results (6.6×10^5 trajectories) for N_0, N_1, and N_2 with an initial seed, $N_1(0)=100$. All quantities plotted in this and subsequent figures are dimensionless. The horizontal axis is a parametrized time, $\xi=\chi|\alpha(0)|^2t$.

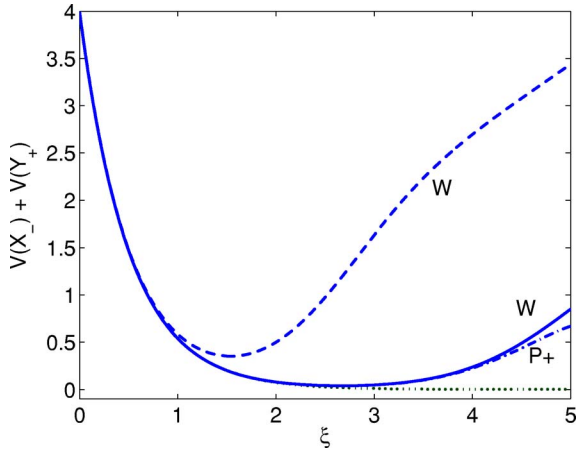

FIG. 2. (Color online) The Duan correlation, $V(\hat{X}_-)+V(\hat{Y}_+)$, with and without an injected seed. The lower three lines are for the spontaneous case, with the dotted line being analytical, the dash-dotted line being the positive-P prediction, and the full line being the Wigner prediction. The dashed line is the Wigner prediction with $N_1(0)=100$. Note that the analytical solution is almost indistinguishable from the positive-P solution until the latter turns and begins to increase.

$$V^{\text{inf}}(\hat{X}_i)V^{\text{inf}}(\hat{Y}_i)=\frac{1}{\cosh^2 2\kappa t}<1 \text{for } t>0, \tag{9}$$

which obviously violate the appropriate inequalities and can be compared to the numerical results obtained below. As can be seen from the figures, this analytic approximation is excellent for short times.

B. Numerical methods

To analyze the system without making the parametric approximation, we will use stochastic integration using the well-known phase-space representations of quantum optics. The Hamiltonian of the system may be mapped exactly onto stochastic differential equations in the positive-P representation [20] following the usual methods [21]. Making the correspondences $\hat{a}_0\to\alpha$, $\hat{a}_1\to\beta$, $\hat{a}_2\to\gamma$, the stochastic equations are found to be

$$\frac{d\alpha}{dt}=-2\chi\alpha^+\beta\gamma+\sqrt{-\chi\beta\gamma}\eta_1,$$

$$\frac{d\alpha^+}{dt}=-2\chi\alpha\beta^+\gamma^+ +\sqrt{-\chi\beta^+\gamma^+}\eta_2,$$

$$\frac{d\beta}{dt}=\chi\alpha^2\gamma^+ +\sqrt{\chi\alpha^2/2}(\eta_3+i\eta_5),$$

$$\frac{d\beta^+}{dt}=\chi\alpha^{+2}\gamma+\sqrt{\chi\alpha^{+2}/2}(\eta_4+i\eta_6),$$

$$\frac{d\gamma}{dt}=\chi\alpha^2\beta^+ +\sqrt{\chi\alpha^2/2}(\eta_3-i\eta_5),$$

$$\frac{d\gamma^+}{dt}=\chi\alpha^{+2}\beta+\sqrt{\chi\alpha^{+2}/2}(\eta_4-i\eta_6), \tag{10}$$

where the η_j are real Gaussian noise terms with the correlations

$$\overline{\eta_j}=0, \quad \overline{\eta_j(t)\eta_k(t')}=\delta_{jk}\delta(t-t'). \tag{11}$$

The solutions for arbitrary normally ordered operator expectation values at time t are found by numerically integrating these equations for a large number of trajectories and forming the appropriate ensemble average via, e.g.,

$$\overline{(\alpha^+)^m\alpha^n}\to\langle:(\hat{a}_0^\dagger)^m\hat{a}_0^n:\rangle, \tag{12}$$

where the overline represents a classical average and the expectation value is normally ordered.

It is of interest to compare the positive-P solutions to those of the approximate, but stable, truncated Wigner representation. This has been used with some success in investigations of BEC [22–28], and allows for the calculation of symmetrically ordered operator moments. Again following standard procedures [21], we can map the system Hamiltonian onto a generalized Fokker-Planck equation for the Wigner pseudoprobability distribution, which has third-order derivatives and hence no equivalent stochastic differential equations. Although methods exist for a mapping onto stochastic difference equations [29], these seldom result in equations that are simple to integrate numerically. We will therefore truncate the third-order terms and map the resulting Fokker-Planck equation onto differential equations for the Wigner variables. This is justified here since the number of particles is much larger than the number of modes. This results in the set of equations

$$\frac{d\alpha}{dt} = -2\chi\alpha^*\beta\gamma, \quad \frac{d\beta}{dt} = \chi\alpha^2\gamma^*, \quad \frac{d\gamma}{dt} = \chi\alpha^2\beta^*. \quad (13)$$

Note that, although these equations may appear deterministic, the initial values are chosen from the appropriate Wigner distribution, so that quantum noise is included in the initial conditions. Symmetrically ordered operator expectation values at time t are found by ensemble averages, e.g.,

$$\overline{(\alpha^+)^m\alpha^n} \to \langle(\hat{a}_0^\dagger)^m\hat{a}_0^n\rangle_{\text{sym}}. \quad (14)$$

C. Numerical results

The numerical solutions are best parametrized by $\xi = \chi|\alpha(0)|^2 t$. For our results we chose an initial coherent state for mode k_0 containing $|\alpha(0)|^2 = 10^4$ atoms, with modes k_1 and k_2 either both unoccupied or one occupied with a small seed of 100 atoms. In practice, we found that integration of the positive-P equations of motion became unstable for times greater than $\xi = 6$, and was probably not trustworthy after $\xi \approx 5$. However, this covers the region of maximum violation of the inequalities of Eqs. (5) and (7). The results are presented in the three figures, which allow us to compare the predictions of the approximate analytic solutions, the formally exact positive-P representation solutions, and those of the truncated Wigner representation. We show the atom numbers in the three modes in Fig. 1 for both the spontaneous and seeded situations. For the spontaneous case, we find that the positive-P and Wigner methods give almost identical results over most of the range shown, thus we can be confident of the Wigner solutions in this regime. We see that a seed in the k_1 mode with only 1% of the number of atoms in the k_0 mode gives appreciably faster scattering into modes k_1 and k_2, with almost full occupation at a time when the spontaneous case has seen approximately 10% of the atoms scattered. A similar effect was observed in the experiment of Campbell *et al.* [4].

We now investigate the quantum correlations between the two scattered modes, in both the spontaneous and seeded cases. In Figs. 2 and 3, we see that the spontaneous process gives an almost complete violation of the inequalities over a relatively large range of interaction times. We also see that, with one of the modes seeded at 1% of the number in the k_0 mode, the inequalities are still strongly violated. This system, therefore, seems more robust to seeding than the nondegenerate optical parametric amplifier (OPA), where, for example, an injection level of 1% had a much less noticeable effect on the mean number of down-converted photons, but was sufficient to almost destroy some quantum correlations [30]. We also note here that the model of the four-wave-mixing process and the OPA are only similar once the parametric approximation is made, which results in a quadratic Hamiltonian for both cases.

It may be of considerable practical interest that the slightly lesser violations occur for large numbers of atoms in the two scattered modes once the system is seeded, as can be seen by comparison with Fig. 1. Also of interest is that the EPR product demonstrates entanglement with seeding in a region where the form of the Duan inequality that we have used does not. This is not a contradiction as the most general form of the Duan inequality [13] does not define the quadratures in the symmetric manner we have used here, so that a more appropriate form would be violated.

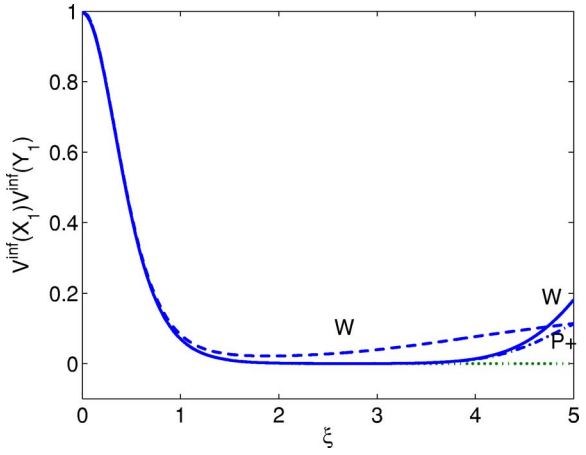

FIG. 3. (Color online) The EPR correlation, $V^{\text{inf}}(\hat{X}_1)V^{\text{inf}}(\hat{Y}_1)$. The dash-dotted line is the spontaneous positive-P result, the solid line is the spontaneous Wigner result, and the dashed line is the stimulated Wigner result with $N_1(0) = 100$. The analytical solution is again indistinguishable from the numerical solutions up until they begin to turn.

IV. CONCLUSIONS

We have calculated the entanglement properties of a Hamiltonian that gives a simplified description of the lattice four-wave-mixing experiment of Campbell *et al.* We have shown that the system exhibits entanglement between the scattered modes and is a candidate for a demonstration of the EPR paradox with massive particles. Seeding of one of the scattered modes allows for substantially quicker conversion than in the spontaneous case, without degrading the violation of the appropriate inequalities significantly. As the entanglement is between the entire modes rather than between individual pairs of scattered atoms, it is not destroyed by small rates of atomic loss, whereas entanglement between individual pairs would be much more sensitive to such losses. A fully quantum spatial analysis that could more closely describe experiments is currently being investigated.

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