Atoms on a Wire: Exact Results

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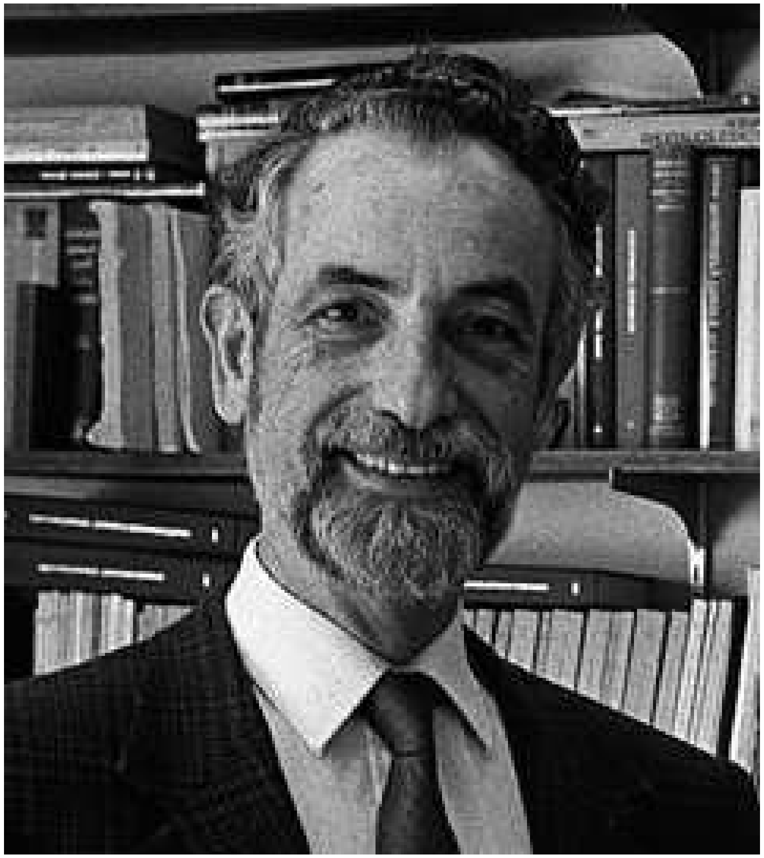
This is not about: *Bird on a Wire*

Outline

- Exact QFT solutions for 1D Bose gas ground state
- Exact solutions for finite temperature 1D Bose gas
- **Exact second-order correlation function**
- Making 1D Bose gases
- How can we measure correlations?
- Summary

Ground states: Girardeau (1960), Lieb (1963)



Girardeau Limit - 1960 - FERMIONIZATION

- Two impenetrable bosons, relative wave-function $\Psi(z_1 - z_2)$.
- Must have $\Psi(0) = 0$.
- Momentum eigenstate, $\Psi(z) = |\sin(kz)|$ **(SYMMETRIC)**.
- Cf identical fermions: $\Psi(z) = \sin(kz)$ **(ANTI-SYMMETRIC)**.
- But $E = \hbar^2 k^2/m$, $\rho = \sin^2(kz)$ in BOTH cases.
- Impenetrable 1D Bose gas $\Leftrightarrow$ non-interacting Fermi gas!

Hamiltonian of a $1D$ Bose gas

Hamiltonian of $1D$ bosons with delta-function interactions

$$\hat{H} = -\sum_j \frac{\partial^2}{\partial z_j^2} + C \sum_{i \neq j} \delta(z_i - z_j)$$

- C – coupling constant ($C > 0$ in BEC)
- Simplified units: $\hbar = 2m = 1$
- General case - particles pass each other

While listening to the Beatles in 1963..

The Beatles first album
-Please Please Me
was a smash hit!

Lieb Solution - 1963

- Repulsive interactions, $C > 0$,
- Slope of the relative wave-function jumps by $2C$ on collision
- Need to construct symmetrized N-body eigenfunction
- General form: $\psi(z_1, ..z_N) = e^{\sum_j ik_j z_j} \; [z_i \neq z_j]$, $E = \sum k_i^2$.
- Uses 'Bethe Ansatz' (Bethe, 1931!!).

Attractive case - quantum soliton

Localized $N-$particle solution, $C < 0$:

$$\psi(z_1, ..z_N) = e^{C\sum_{i>j}|z_i - z_j|/2},$$

Ground-state energy (Lieb, 1963; McGuire, 1964):

$$E = -\frac{N(N^2-1)}{12}C^2$$

Observation:

- Corresponds to soliton in optical fibre
- Quantum effects small, since $N \simeq 10^9$
- Interactions cause correlations of photons
- Quantum squeezing predicted (Carter, Drummond et al 1987)
- Required quantum dynamical calculation
- Observed experimentally at IBM, 1992

Repulsive case - 1D BEC

- $\psi(z_1,..z_N) = \prod_{i=1}^{N} e^{ik_i z_i} \left[\prod_{j>i}\left(1 - \frac{iC}{k_i - k_j}\varepsilon(z_i - z_j)\right)\right],$
- Here $\varepsilon(z_i - z_j)$ is the sign function, k_i is the 'quasi-momentum'.
- $k_{j+1} - k_j$ determined by the boundary conditions at $z_i = z_j$.
- $kL = 2\pi n_k - 2\sum_{k'} \tan^{-1}([k - k']/C)$
- Energy eigenvalue: $E = \sum k_i^2$.

Eigenvalue?

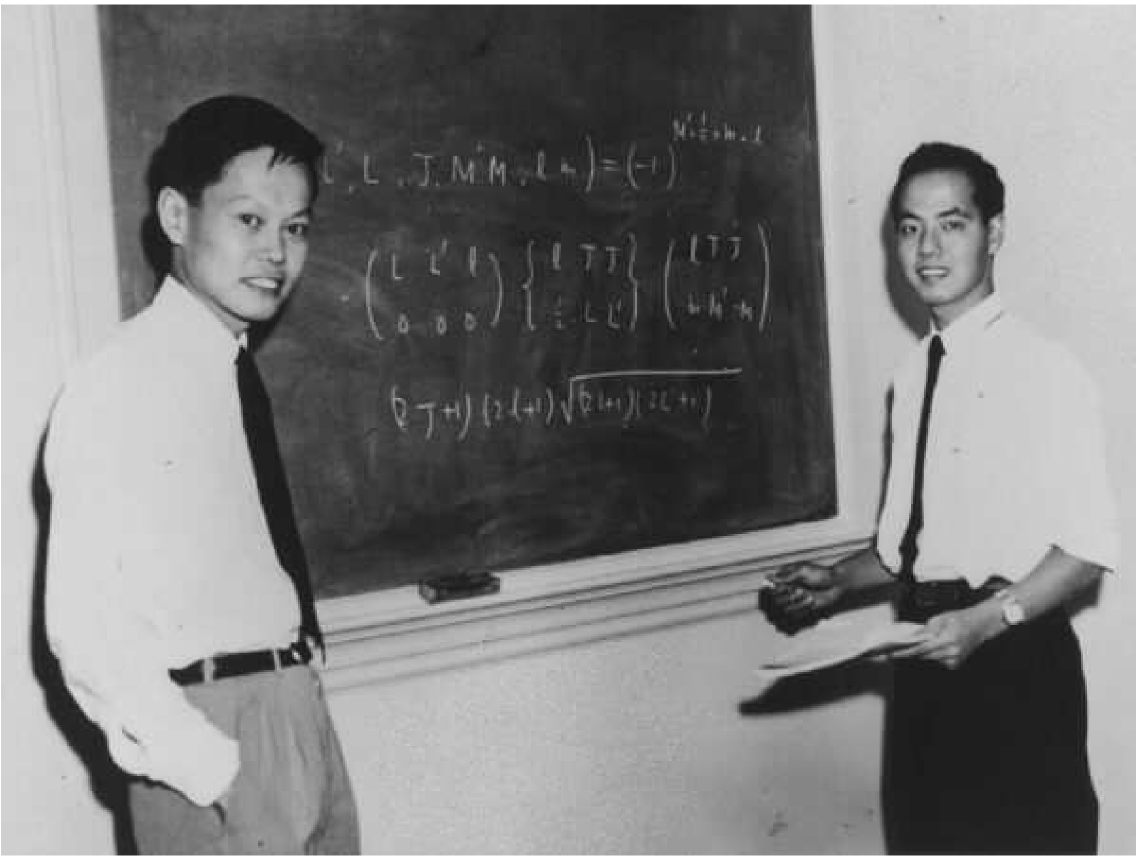
Energy from integrating the quasi-momentum density.

For a large system $L \to \infty$, quasi-momenta spacing $\to 1/Lf(k)$ where:

$$2\pi f(k) = 1 + 2C \int_{-K}^{K} \frac{f(p)dp}{C^2 + (p-k)^2}$$

Hence, the energy is: $E = L \int_{-K}^{K} f(k) k^2 dk$

Finite Temperatures: Yang & Yang, 1967



(This is Yang & Lee,
Nobel Laureates
for parity violation!)

Holes in the Lieb solution..

- C. N. and C. P. Yang worked on the finite-temperature solution.
- Excited states are similar to the ground state,
- Some of the quasi-momenta are ‘missing’ - like holes.
- This adds entropy to the system.

Yang-Yang equations

At thermal equilibrium:

$$2\pi f(k)\left[1+e^{\varepsilon(k)/T}\right]=1+2C\int\frac{f(p)dp}{C^2+(p-k)^2}$$

where: $\varepsilon(k)$ is calculated from a second integral equation:

$$\varepsilon(k)=-\mu+k^2-\frac{TC}{\pi}\int\frac{\ln\left[1+e^{-\varepsilon(p)/T}\right]dp}{C^2+(p-k)^2}$$

Result

- **Results scale with dimensionless coupling:** $\gamma = C/\rho$
- COUPLING IS STRONGEST AT LOW DENSITY (ρ)!
- **Reduced temperature:** $\tau = T/T_Q$,
- Here $T_Q = \rho^2$ is the temperature of quantum degeneracy, in energy units ($k_B = 1$).

Correlation function

- What can we observe to test this theory?

- RELATIVE PROBABILITY THAT TWO ATOMS ARE DETECTED AT THE SAME z:

- $$g^{(2)} \equiv \frac{\langle \hat{\Psi}^{\dagger}\hat{\Psi}^{\dagger}\hat{\Psi}\hat{\Psi} \rangle}{\rho^2}$$

- No finite temperature solution in 40 years!!

CANONICAL METHOD

Using the Hellmann-Feynman theorem:

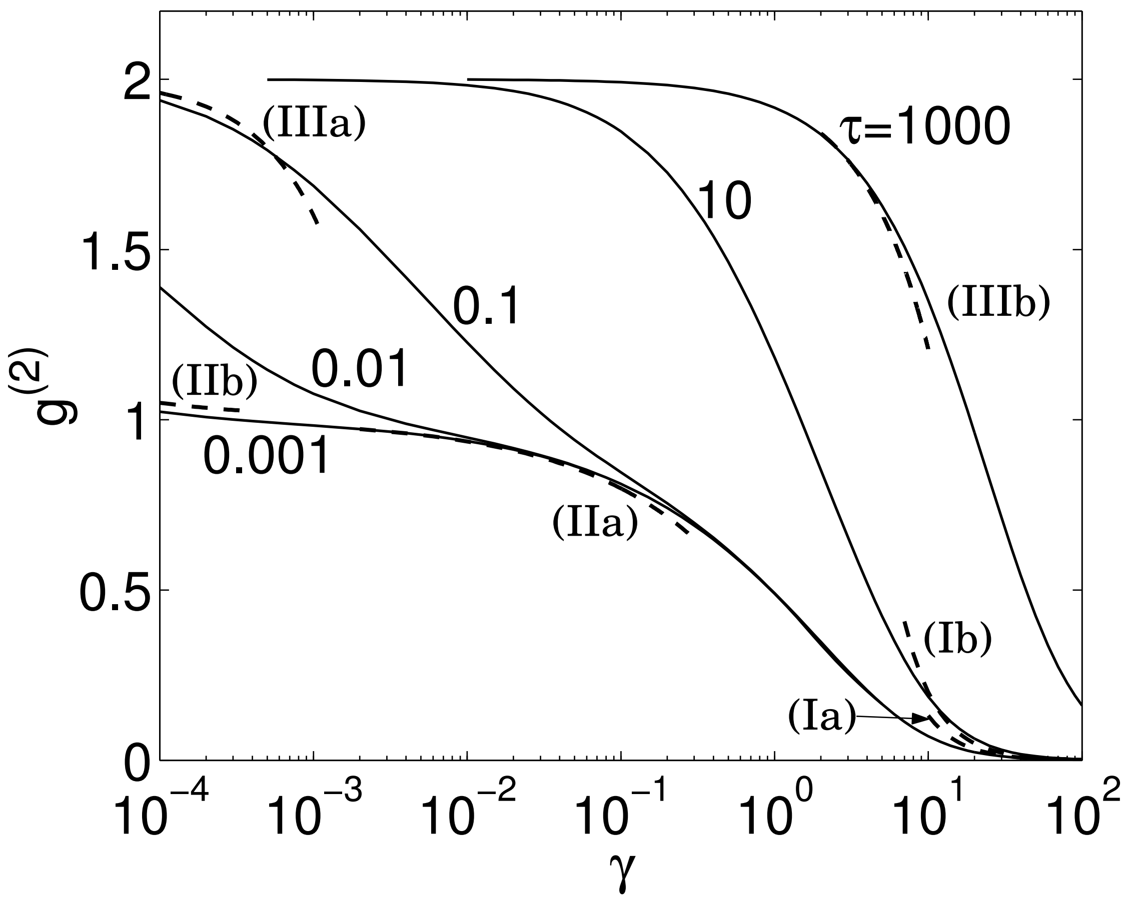
$$\frac{\partial F}{\partial C} = -T \frac{\partial \log Z}{\partial C} = \frac{1}{Z} \mathrm{Tr} \left[\frac{\partial \hat{H}}{\partial C} \exp(-\hat{H}/T) \right]$$

Normalized two-particle correlation function :

$$g^{(2)} \equiv \frac{\langle \hat{\Psi}^\dagger \hat{\Psi}^\dagger \hat{\Psi} \hat{\Psi} \rangle}{\rho^2} = \frac{1}{N\rho^2} \left(\frac{\partial F(\gamma, \tau)}{\partial \gamma} \right)_{\rho,\tau}$$

$F(\gamma, \tau)$ **can be calculated exactly, by numerically solving the Yang-Yang integral equations**

SOLUTION



2
1.5
1
0.5
0
$g^{(2)}$
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
γ
(IIIa)
τ=1000
10
0.1
(IIIb)
0.01
(IIb)
0.001
(IIa)
(Ib)
(Ia)

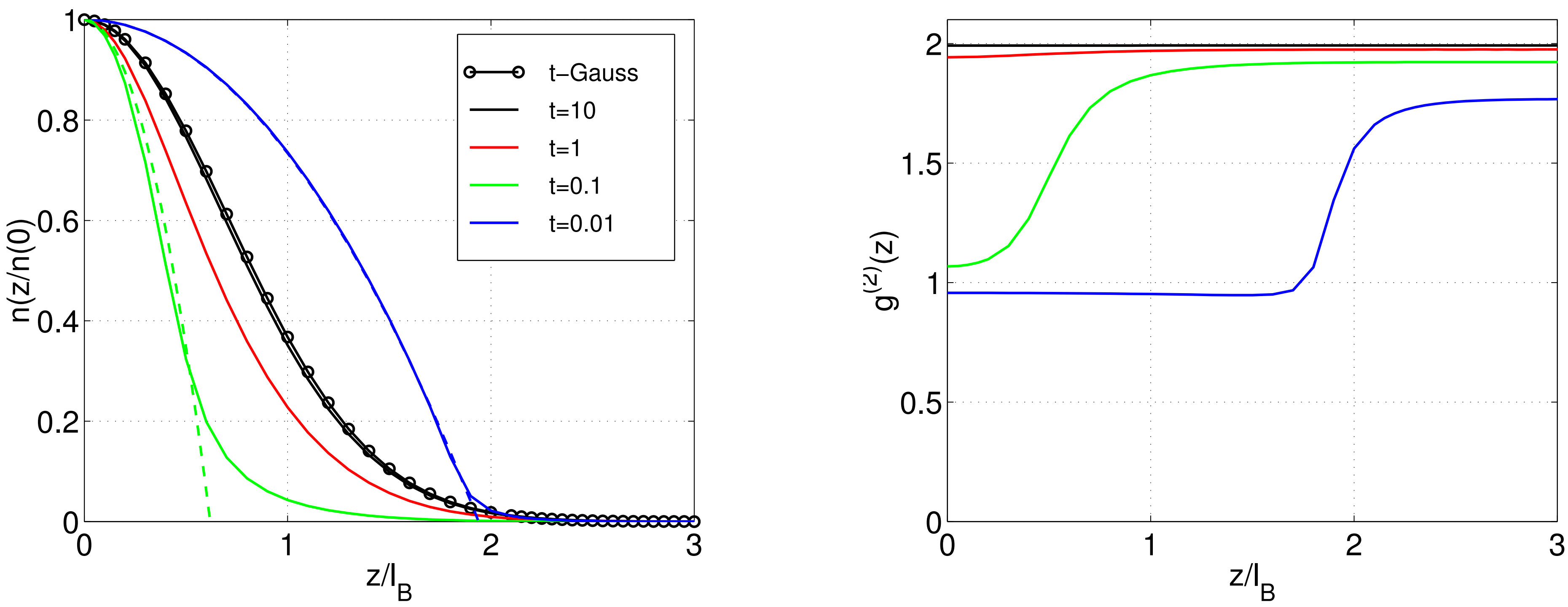
1D Bose gas in a trap

Local density approximation: For a large trap, use a local effective chemical potential, assume uniform solution holds locally with

$$\mu(z) = \mu_0 - V(z) = \mu_0 - \omega_z^2 z^2/4,$$

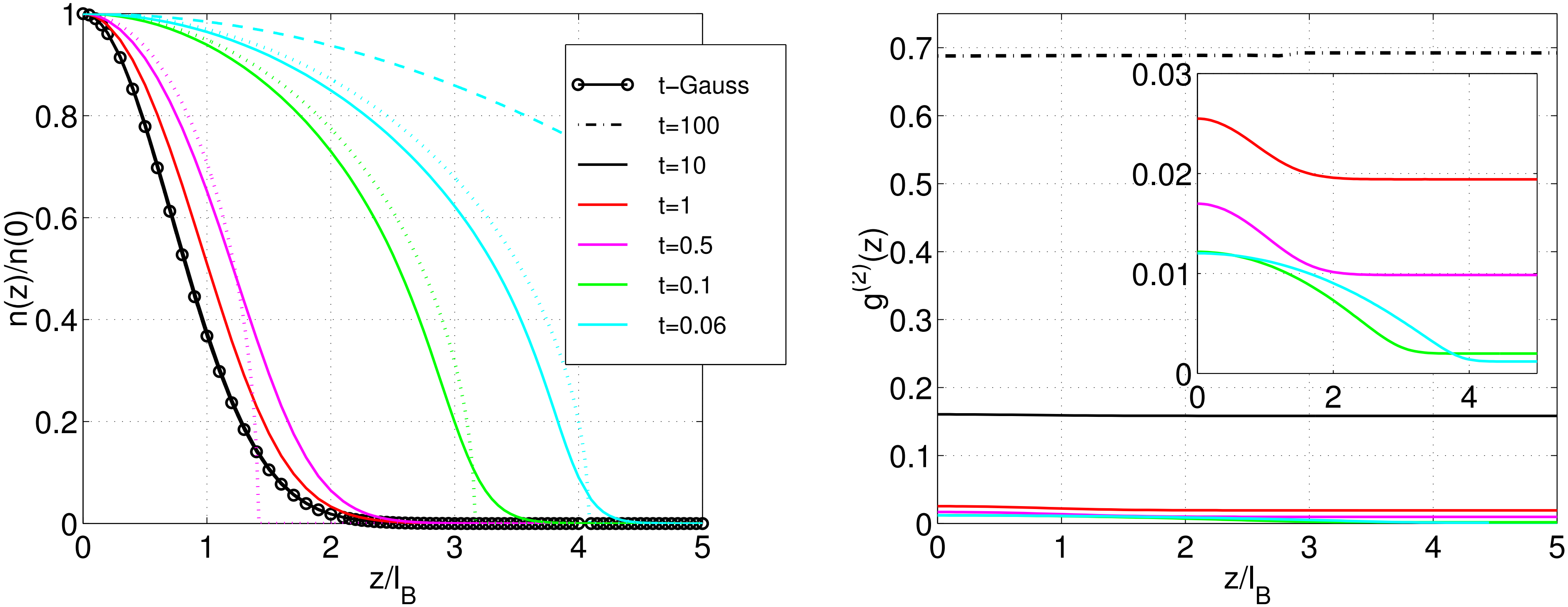
- Relative temperature: $t = T/T_Q = T/(N\omega_z)$,
- Interaction parameter: $\tilde{\gamma} = C\sqrt{1/(N\omega_z)}$
- Classical radius: $l_B = \sqrt{4T/(\omega_z^2)}.$

Weakly interacting (GP) regime ($\tilde{\gamma} = 0.01$)



DASHED LINE IS THOMAS-FERMI DENSITY PROFILE

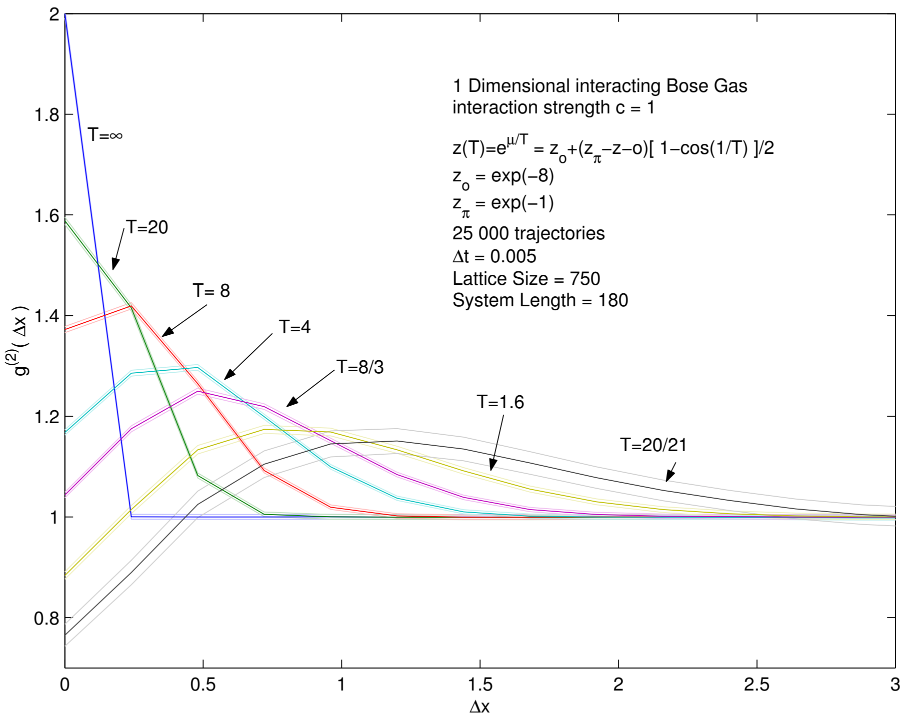
Strongly interacting (TG) regime ($\tilde{\gamma} = 10$)



DOTTED LINE IS GIRARDEAU DENSITY PROFILE

Spatial correlations

So far, no exact solution, but can be calculated using stochastic methods:



Making a 1D Bose gas

1D regime REQUIRES: $a \ll l_\perp \ll \{1/\rho, \Lambda_T\}$,

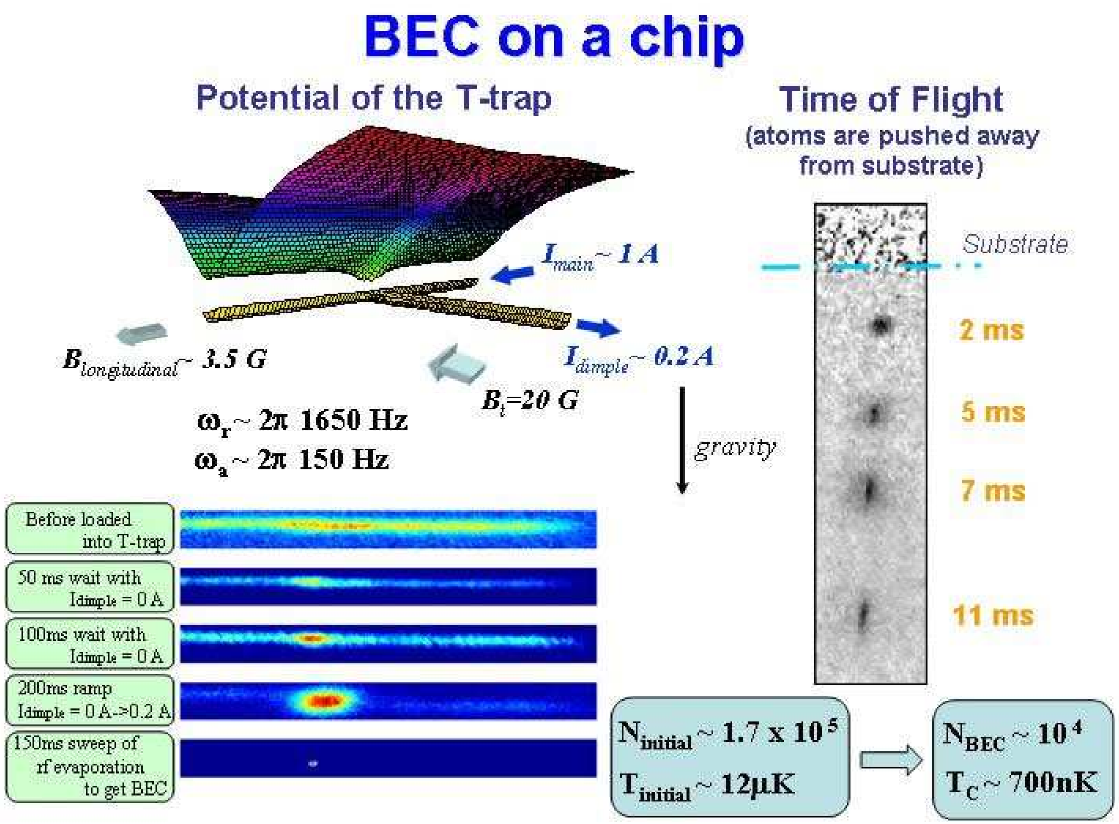
where:

- $\Lambda_T = (4\pi/T)^{1/2}$ – thermal de Broglie wavelength, – ρ – 1D (linear) density.
- $l_\perp = \sqrt{2/\omega_\perp}$ – amplitude of transverse zero point oscillations.
- $C = 2a/(l_\perp^2)$ – coupling constant

Experimental techniques

- Laser waveguiding: use a red detuned laser to attract atoms into a cylindrical region (MIT)
- Lattices: use an optical lattice in 2-D to make a 'crystal' of 1-D optical wires(NIST)
- Atom chips: use a wire on a surface to create a 1D magnetic trap(UQ!)
- Video tape: use permanent magnetic domains on a surface (Harvard, Swinburne).

Example - NIST T-trap



Measuring correlations

- Photo-association - molecule formation depends on $g^{(2)}$
- Feshbach resonance in 1D depends on $g^{(2)}$?
- Ramsey fringe spectroscopy - shown at NIST in 3D
- Direct detection - integrated wire electrodes, field ionization

SUMMARY

- 1D Bose correlations at finite temperature exactly soluble
- Several different physical regimes
- Weak coupling and high-density interchangeable!
- Weak coupling→large fluctuations: $g^{(2)} \to 2$
- Intermediate coupling →coherent behaviour: $g^{(2)} \to 1$
- Strong coupling →fermionic behaviour: $g^{(2)} \to 0$

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