How to mode-lock an atom laser

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Summary

- We consider a possible technique for modelocking an atom laser.
- The laser is constructed as a torus using magnetic trapping techniques.
- Modelocking is achieved by using a periodic output coupler to generate a dark soliton with angular momentum of $\hbar/2$.
- This results in a periodically replicating coherent field inside the torus.
- The soliton is started by laser stirring
- **The mode-locking process leads to a stable dark soliton with fractional angular momentum quantum number.**

Introduction

The discovery of Bose-Einstein condensation (BEC) in ultra-cold alkali-metal vapors at temperatures of $10^{-6}K$ or lower, has raised the possibility of a coherent atom laser. Recent experimental developments show that this is, indeed, practical.

We suggest the idea of making use of the atom-atom repulsion to form a toroidal soliton - which can then form a reproducible, coherent waveform. This allows the creation of a modelocked, pulsed atom laser.

- **The advantage of modelocking in lasers, is that the sequence of output pulses are in phase with each other.**

Gross-Pitaevskii equation

In the limit of a deep trap, we suppose that any mode-structure transverse to the trap circumference can be neglected.

We assume the trapped atoms are not sufficiently hot and energetic to excite any transverse mode except the lowest lying mode.

- **This is the principle limitation of the present treatment, and requires a relatively small torus, with a deep trapping potential, and low temperatures.**

One-dimensional G-P equation

The condensate wave-function obeys the one-dimensional Gross-Pitaevskii (G-P) equations, in a 1D ring of radius R. The wave function is $\Psi(\theta, t)$, where θ is the angle around the ring, so:

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2mR^2} \frac{\partial^2 \Psi}{\partial \theta^2} + U(\theta) + u|\Psi|^2 \Psi \, , \tag{1}$$

where the coupling constant u is given by

$$u = \frac{4\pi\hbar^2 a}{m} \, , \tag{2}$$

Here a is the scattering length, while m is the atomic mass, u is the inter-particle effective potential, and U is the single-particle potential due to trap non-uniformity.

Normalization condition

The normalization condition is obtained from noting that the condensate wave function gives the effective particle density:

$$\int_0^{2\pi} d\theta \Psi^* \Psi = \frac{N}{AR} \,. \tag{3}$$

Here N is the number of particles, and A is the effective mode cross-sectional area, transverse to the circumference of the torus. The boundary condition is, from simple periodicity requirements

$$\Psi(\theta + 2\pi, t) = \Psi(\theta, t) \,. \tag{4}$$

Hamiltonian

The Hamiltonian of the system that would classically generate the G-P equation, is:

$$H = AR\int_0^{2\pi} d\theta \left[\frac{\hbar^2}{2mR^2}\frac{\partial\psi^*}{\partial\theta}\frac{\partial\psi}{\partial\theta} + \frac{u}{2}\left(\psi^*\psi\right)^2 + U(\theta)\psi^*\psi\right] \qquad (5)$$

- **The quantized version of this Hamiltonian is the problem of a one-dimensional repulsive Bose gas with periodic boundary conditions.**
- **This has well-known exact eigenstates at the quantum level.**

Dimensionless variables

We define dimensionless variables by setting:

$$
\begin{aligned}
\alpha &\equiv \frac{NmR}{\pi\hbar^2 A}u = \frac{4NRa}{A}\,, \\
\tau &\equiv \frac{\hbar}{2mR^2}t\,, \\
V(\theta) &= \frac{2mR^2}{\hbar^2}U(\theta) \\
\psi(\theta,\tau) &\equiv \Psi(\theta,t)\sqrt{\frac{2\pi AR}{N}}\,.
\end{aligned}
\tag{6}
$$

Dimensionless G-P equation

This gives a dimensionless form of the G-P equation:

$$i\frac{\partial\psi}{\partial\tau} = -\frac{\partial^2\psi}{\partial\theta^2} + V\psi + \alpha|\psi|^2\psi\,. \tag{7}$$

The normalization and periodicity conditions are now:

$$\begin{aligned} \int_0^{2\pi} d\theta\psi^*\psi &= 2\pi\,, \\ \psi(\theta+2\pi,\tau) &= \psi(\theta,\tau)\,. \end{aligned} \tag{8}$$

Conserved Quantities

We can define a dimensionless energy per particle:

$$W \equiv \frac{2mR^2}{\hbar^2 N} H = \frac{1}{2\pi} \int_0^{2\pi} d\theta \left[\frac{\partial \psi^*}{\partial \theta} \frac{\partial \psi}{\partial \theta} + \frac{\alpha}{2} \left(\psi^* \psi \right)^2 \right] . \tag{9}$$

The dimensionless angular momentum per particle is given by:

$$J = \frac{1}{4\pi i} \int_0^{2\pi} d\theta \left(\psi^* \frac{\partial \psi}{\partial \theta} - \psi \frac{\partial \psi^*}{\partial \theta} \right) \tag{10}$$

Nonrotating solutions

For a uniform state, in the case of repulsive interaction ($\alpha > 0$), and zero external potential, we must clearly have (by the normalization condition):

$$\psi^* \psi = 1 \,. \tag{11}$$

The lowest energy solution is just the uniform condensate, which evolves in time according to:

$$\psi_0(\theta, \tau) = e^{-i\nu_0 \tau} \,, \tag{12}$$

Uniform vortex

A uniform vortex state with angular momentum l has the condensate wave function

$$\psi_l(\theta, \tau) = e^{-i\nu_l \tau + il\theta} \tag{13}$$

where l must be an integer, in order to satisfy the boundary condition. The equation of motion gives

$$\nu_l = \alpha + l \, . \tag{14}$$

The energy and momentum per particle of the uniform vortex state are

$$\begin{aligned} W &= \alpha/2 + l^2 \, , \\ J &= l \qquad\qquad (l = 1, 2, 3, \cdots) \, . \end{aligned} \tag{15}$$

Soliton solutions

The one-dimensional NLSE is known to be classically integrable, and to have periodic soliton solutions that correspond to localized disturbances traveling around the circumference of the torus. For a soliton solution, put

$$\psi(\theta, \tau) = f(\theta - v\tau)e^{i\eta(\theta,\tau)}\,, \tag{16}$$

where v is an arbitrary real parameter, corresponding to the soliton velocity. In general, a continuous set of velocities is possible. Complex values of f correspond to grey solitons, having non-integer velocities.

- **Real values of f correspond to dark and bright solitons, having integer values of v.**

Dark and Bright Solitons

In the case of a real envelope function the boundary condition demands

$$\begin{aligned} f(\bar{\theta}+2\pi) &= (-1)^{2l} f(\bar{\theta}), \\ \eta(\theta+2\pi,\tau) &= \eta(\theta,\tau)+2\pi l, \end{aligned} \tag{17}$$

where $\bar{\theta} \equiv \theta - v\tau$, and $l = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 2, \cdots$. Assume the form

$$\eta(\theta,\tau) = l(\theta - v\tau) - \omega\tau, \tag{18}$$

where ω is an arbitrary real parameter. This satisfies the boundary condition. For f to be real, the angular momentum is

$$J = l = v/2. \tag{19}$$

Exact Dark Soliton Solutions

We consider the case $\alpha > 0$.

The explicit solution in terms of the Jacobi elliptic function is:

$$f(\bar{\theta}) = f_2 sn\left(f_1\sqrt{\frac{\alpha}{2}}\left(\bar{\theta} - \pi/s\right)\right) . \tag{20}$$

- **This is an exact solution, with half-integer angular momentum.**

Tanh with a twist

For large α , we find the approximate solution

$$f(\bar{\theta}) = \sqrt{\frac{\beta}{\alpha}} \tanh\left(\sqrt{\frac{\beta}{2}}\,(\bar{\theta} - \pi/s)\right) . \tag{21}$$

Where:

$$\beta \simeq \alpha \left(1 + \frac{1}{2\pi}\sqrt{\frac{2}{\alpha}}\right) . \tag{22}$$

- **To form the solution, the sign change in the tanh is compensated for by an ODD number of π phase-rotations or ‘twists’ due to the condensate orbital angular momentum.**

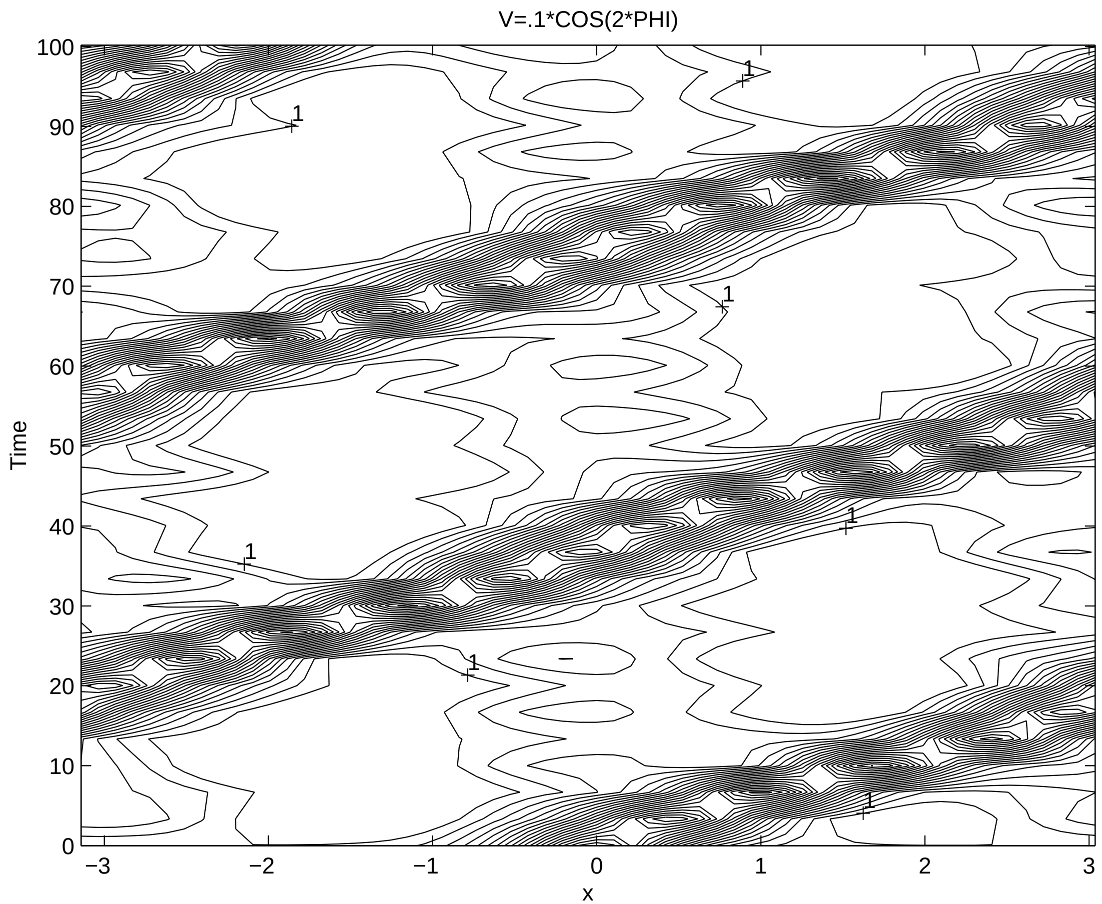
Conserved Soliton Quantities

We can now estimate values of the conserved quantities. For narrow solitons:

$$
\begin{aligned}
W &\simeq \frac{\alpha}{2} + l^2 + \frac{\sqrt{2\alpha}}{3\pi} + O(1)\,, \\
J &= l \qquad (l = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots)\,.
\end{aligned}
\tag{23}
$$

- **While this has a higher energy than the uniform state, the kink singularity is stable in a perturbed environment where rotational symmetry is broken ($V(\theta) = 0.1\cos(2\theta)$), as shown in Fig (1).**

Stability in a non-symmetric trap



Mode-locked atom laser

We now include a stirring potential, gain and loss terms into the G-P equation:

$$i\frac{\partial \psi}{\partial \tau} = -\frac{\partial^2 \psi}{\partial \theta^2} + \alpha|\psi|^2\psi + V\psi + i(g-\gamma)\psi\,, \tag{24}$$

where V is the 'stirring' potential.

- **The gain mechanism g can be provided via stimulated emission from non-condensed atoms that are continuously loaded into the trap.**

- **The loss mechanism γ is due to an output coupler, like a local Raman-tuned transition to a non-trapped state of the atoms.**

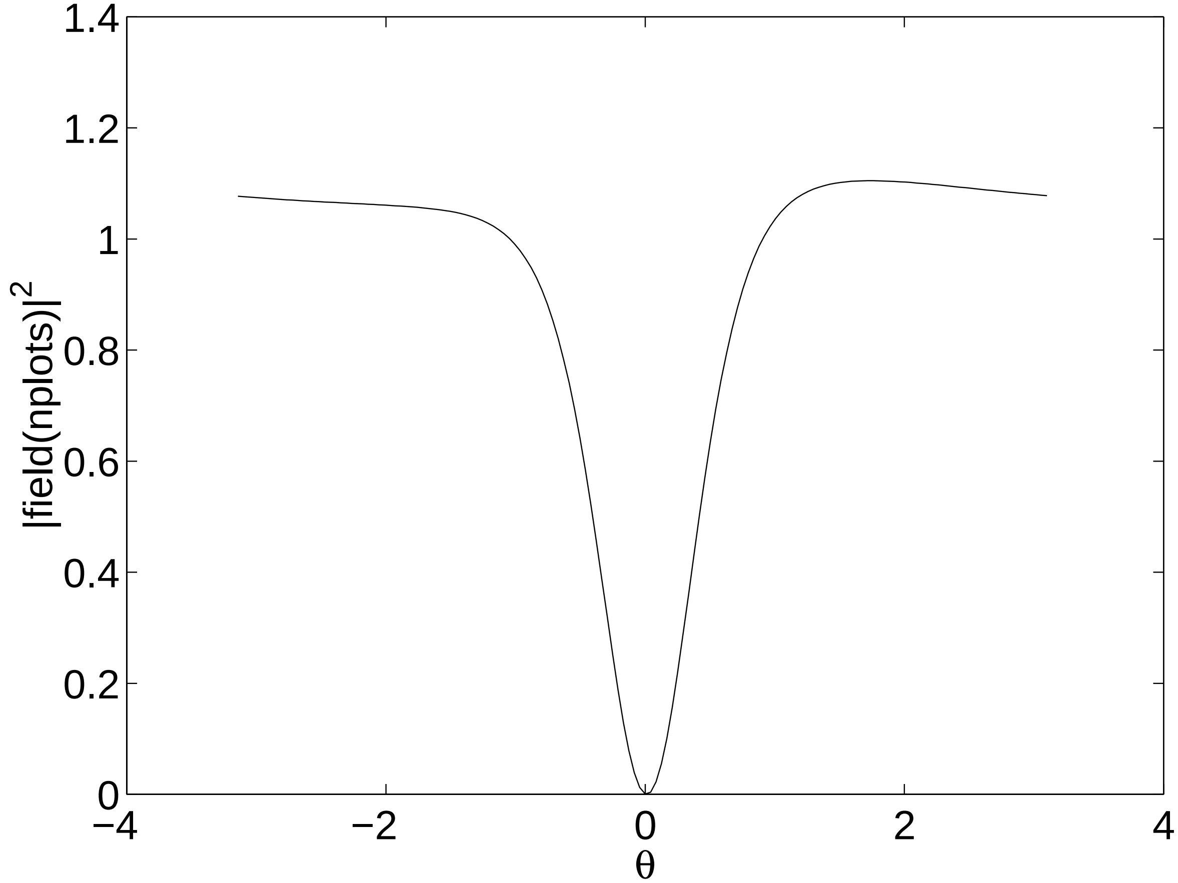
Stabilisation

If the mean atom number in the condensate increases, the hole shrinks in diameter, thus increasing the out-coupling efficiency and losses. On the other hand, a decrease in atom number will increase the hole diameter. This reduces the output coupler efficiency, and therefore decreases the losses.

The same mechanism also gives rise to a stable condensate phase and velocity, which otherwise could vary, allowing a grey soliton to form by continuous deformation of the initial dark soliton phase.

- **The mean atom number in the entire BEC is stabilized by the fact that the output-coupling efficiency is a nonlinear function of the hole diameter, and hence of the atom number, as shown in Fig. (2).**

Stable mode-locked dark soliton

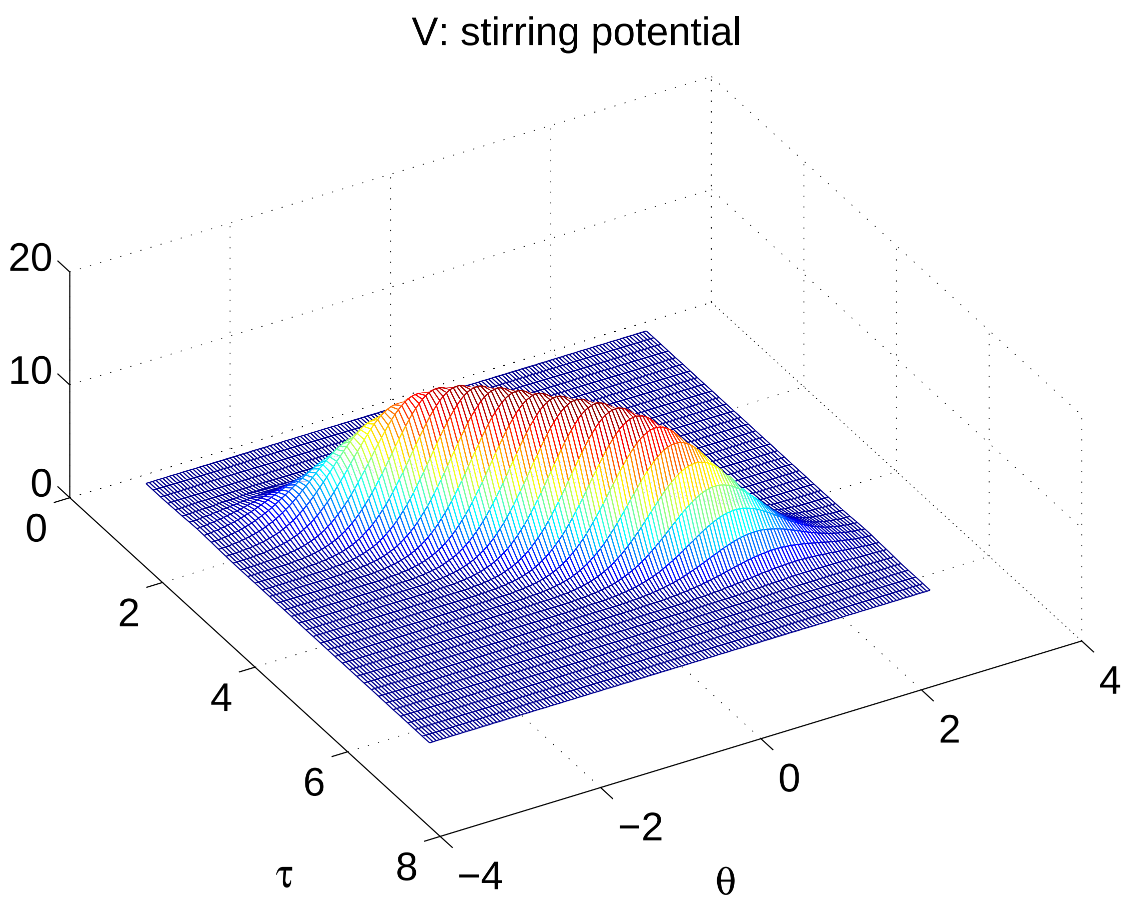


Soliton generation

- We start by assuming that a blue-detuned 'stirring' laser is employed to form a moving, positive potential hill.
- This repels the condensate, forming the phase singularity.
- The potential hill is moved, accelerating the condensate in the required direction, to form a moving tanh-like excitation.
- To give definite results, consider the stirring potential in Fig. (3), chosen to be localized in time, but moving at the desired soliton velocity:

$$V(\theta,\tau) = V_0 e^{-(\pi+\theta-\tau)^2/\sigma_1^2} e^{-(\tau-\pi)^4/\sigma_2^4} . \tag{25}$$

Stirring potential



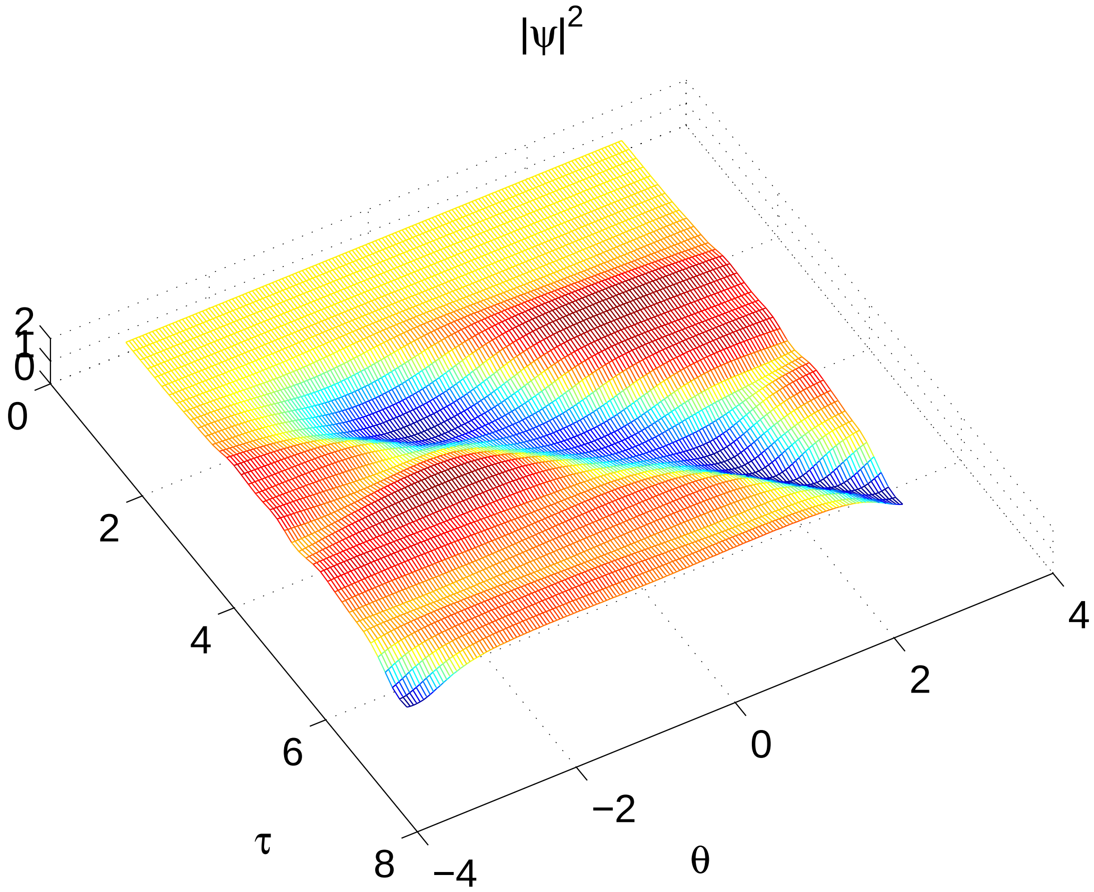
Shaken, not stirred?

In the simulations we assume $\beta = 8$, corresponding to $\alpha = 7.3634$, and choose the values of parameters as follows: $V_0 = 1.673\alpha$, $\sigma_1 = 1.5 \times 1.05 \times \sqrt{2/\beta}$, $\sigma_2 = \pi/2$.

The initial field is chosen to correspond to a uniform condensate: $\psi(\theta, 0) = 1$.

- **The immediate consequence of the stirring laser is shown in Fig. (4), which demonstrates the formation of a highly excited, yet soliton-like structure, but with additional phonon modes excited.**

Stirred Field



Dark Soliton Modelocking

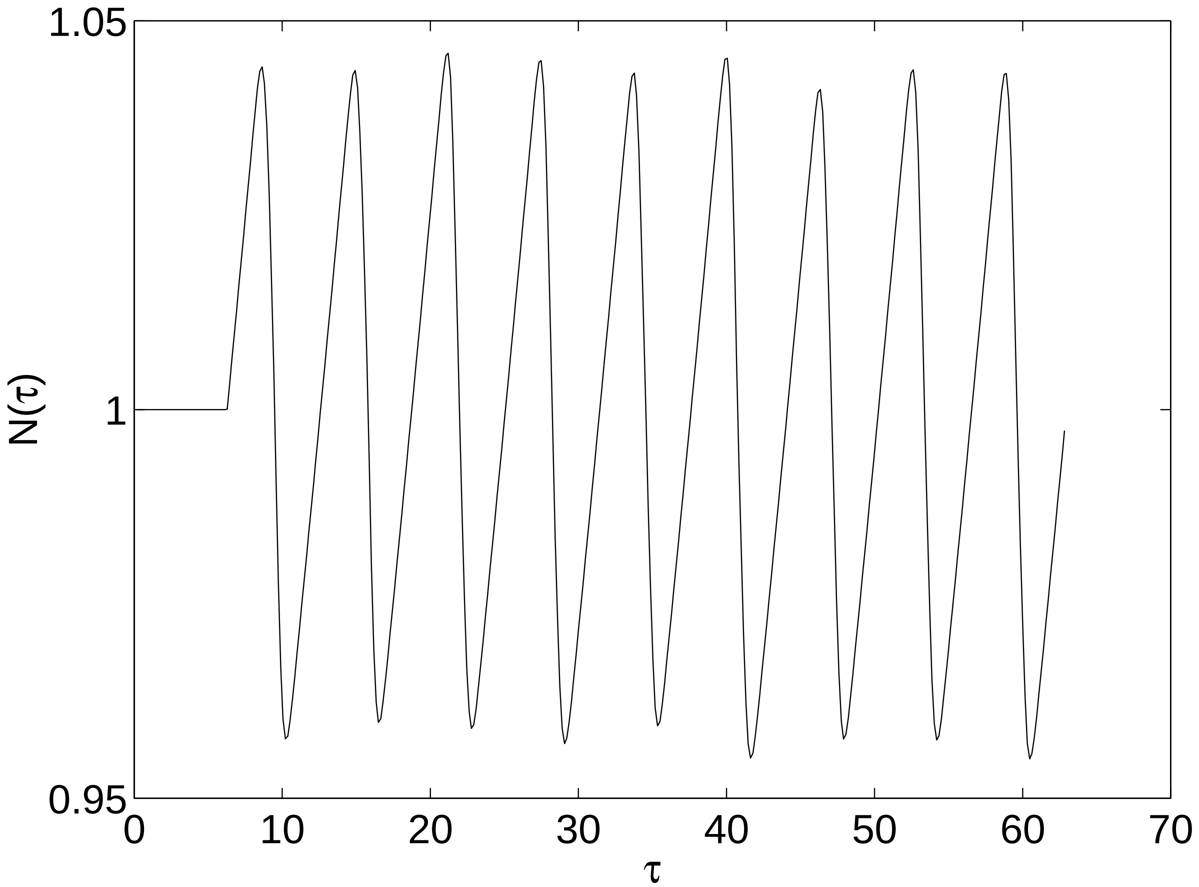
Next, the mode-locking laser can be used to 'clean-up' or stabilize the dark soliton, while also performing the important role of producing an output pulse-train of coherent pulses, as shown in Fig. (5).

We assume the gain is uniform ($g = const = 0.01$), while the loss coefficient $\gamma \equiv \gamma(\theta, \tau)$ is chosen in the following form:

$$\gamma(\theta, \tau) = \gamma_0 e^{-\theta^2/\sigma_\theta^2} \sum_n e^{-(\tau-\pi-nT)^2/\sigma_\tau^2} . \tag{26}$$

where T is a characteristic period, and $n = 1, 2, 3, ...$

Stabilization process



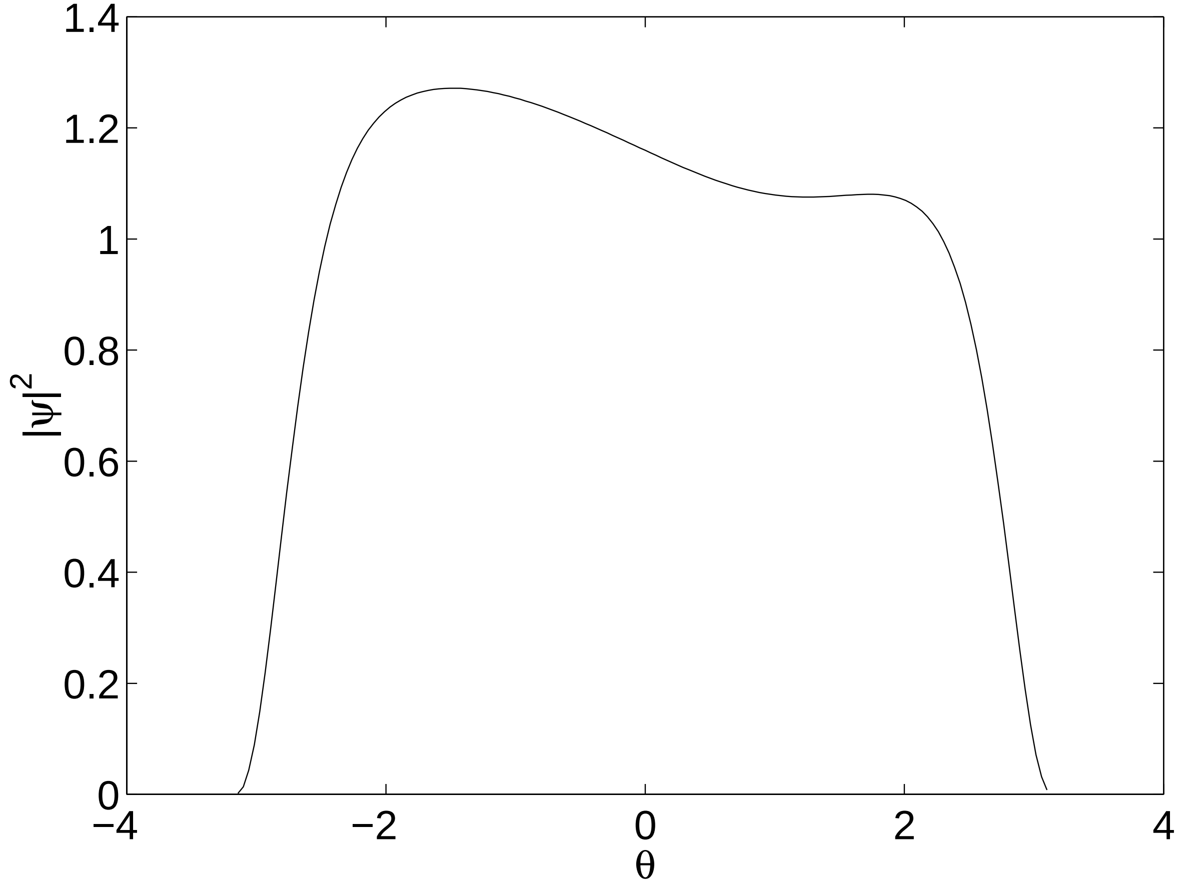
Simulations

In the simulations we assume $\beta = 8$, corresponding to $\alpha = 7.3634$, and choose the values of parameters as follows: $\gamma_0 = 0.9$, $\sigma_\theta = 1.05\sqrt{2/\beta}$, $\sigma_\tau = 1.05\sqrt{2/\beta}$ and $T = 2\pi$.

The results can be clearly seen in Fig. (6).

- **The initially excited soliton gradually stabilizes via the mode-locking output coupler, and forms a stable, self-replicating condensate wave-function inside the atom laser.**

Stirred Field



Conclusion

- The calculations given here demonstrate how to generate stable, circulating dark solitons, in a toroidal atom laser .
- Lasing is achieved by using the super-saturated gain model.
- A periodic output coupler based on Raman transitions is used.
- The result is the generation of a dark soliton with angular momentum of $\hbar/2$ per atom.
- We have given a numerical treatment of soliton generation.
- **A stable, mode-locked Bose-Einstein condensate with a fractional angular momentum quantum number is possible.**