Extending the realms of numerical stochastic integration

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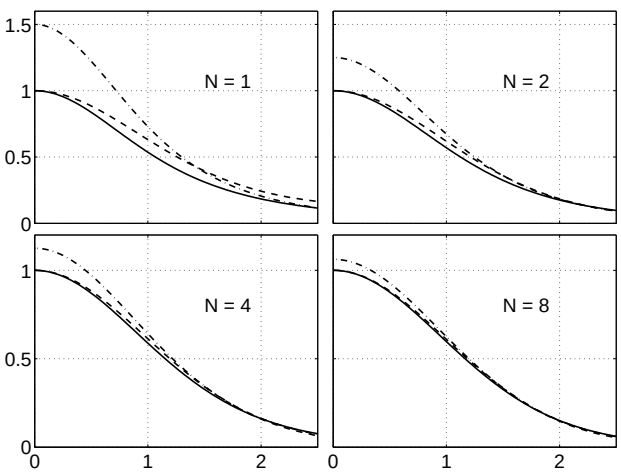
The numerical integration of stochastic differential equations using phase-space representations is a powerful tool in the investigation of interacting multi-body quantum systems. For quantum optics and quantum atom optics, the representations of choice have been the positive-P and truncated Wigner representations. Both these have well known advantages and disadvantages and we have continued our work to strengthen the former and minimise the latter.

As with all numerical work, initial conditions must be specified. When we wish to investigate quantum dynamics, knowing the quantum state of the fields at the initial time can be crucially important. Traditionally, investigations have often begun with coherent initial states, which may be a good approximation for quantum optical systems, but not for ultra-cold quantum gases. Together with Ashton Bradley (now at the University of Otago), methods to numerically simulate a number of different initial quantum states were developed [1]. Apart from an approximation used to model Fock states in the Wigner representation, necessary because the Wigner function for this state takes on negative values, these were all exact. The Wigner approximation for a Fock state becomes more accurate as the occupation number increases, and is already a good approximation for $N = 10$, below which other methods are more appropriate.

For numerical investigations of ultra-cold atoms, where the interaction terms can dominate, the truncated Wigner representation is often the method of choice. While not exact, it leads to tractable stochastic equations which naturally give symmetrically ordered operator expectation values. This is not a problem until we wish to calculate time-normally ordered quantities such as

$$g^{(1)}(\tau) = \langle \hat{a}^\dagger(\tau)\hat{a}(0)\rangle. \tag{1}$$



Two-time correlation functions, $g^{(1)}(\tau)$, for the Kerr oscillator. The solid lines are the exact solutions, the dashed lines are our corrected Wigner solutions and the dash-dotted lines are uncorrected Wigner predictions. The horizontal axis is dimensionless time.

In a collaboration with universities in Ulm, Kaiserslautern and Boston [2], a time-symmetric ordering of the creation and annihilation operators was developed which allows for the calculation of this class of averages. For two-time averages, Kubo's theorem relating the linear response function to two-time commutators is used to calculate the normally-ordered correlation functions. The results have been tested for both the anharmonic (Kerr) oscillator and one dimensional Bose-Hubbard chains with up to ten sites. The method was found to be very accurate and comparatively cheap computationally.
This work can also be combined with recent extensions of the Wigner representation by ACQAO researchers at ANU to add another powerful tool to the arsenal of theoretical quantum atom-optics.

References

[1] M. K. Olsen and A. S. Bradley, Opt. Commun **282**, 3924 (2009).
[2] B. Berg, L. I. Plimak, A. Polkovnikov, M. K. Olsen, M. Fleischhauer, and W. P. Schleich, Phys. Rev. A **80**, 033624 (2009).