(More Australians choose)

Theoretical Atom Optics

(than any competing brand)

Lectures for the ACQAO Summer School

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"An atom laser beam is a coherent beam of atoms"

" 9/243 at lunch (Lehmann 2/42)"

Review

Complete description of multiple bosonic atoms

– The basis states of the total Hilbert space have a given occupation number for each single particle mode:

$$|\psi\rangle = \sum_{n_1,n_2,n_3,\cdots} f_{n_1,n_2,n_3,\cdots} |n_1,n_2,n_3,\cdots\rangle$$

Typical Hamiltonian:

$$\hat{H} = \int d\mathbf{x}\, \hat{\psi}^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x}) \right) \hat{\psi}(\mathbf{x}) + U \int d\mathbf{x}\, \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}(\mathbf{x}) \hat{\psi}(\mathbf{x})$$

These field operators look familiar

Their Heisenberg equation of motion is *very* familiar:

$$i\hbar \frac{\partial \hat{\psi}(\mathbf{x})}{\partial t} = \left(-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x}) + U \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}(\mathbf{x}) \right) \hat{\psi}(\mathbf{x})$$

Don't be fooled - the field operators are *not* the state

- However, they (and their conjugate) can build any operator
- Knowing them is enough to calculate any observable

Q. How would you find the momentum density from $\hat{\psi}(\mathbf{x})$?

Major classes of approximation

Quantum field theory calculations are hard

Approximations ignore either:

– Complexity in the quantum state of each mode

– The number of modes

– Systems with strong interactions

Perturbative methods: incredibly refined in the world of QED, some branches of condensed matter physics and particle physics. Unfortunately, atom optics is typically highly non-perturbative

Semiclassical approximation

Ignore complexity in the quantum state

$$|\psi\rangle = \sum_{n_1,n_2,n_3,\cdots} f_{n_1,n_2,n_3,\cdots} |n_1,n_2,n_3,\cdots\rangle$$

1. Assume it factorises

$$\approx \sum_{n_1,n_2,n_3,\cdots} c_{n_1} c_{n_2} c_{n_3} \cdots |n_1,n_2,n_3,\cdots\rangle$$

$$= \bigotimes_j \sum_{n_j} c_{n_j} |n_j\rangle$$

2. Assume each state is a coherent state

$$\hat{b}_j |\beta_j\rangle = \beta_j |\beta_j\rangle \qquad |\beta_j\rangle = \sum_{n_j} e^{-\frac{|\beta_j|^2}{2}} \frac{\beta_j^{\,n_j}}{\sqrt{n_j!}} |n_j\rangle$$

$$\boxed{|\psi\rangle \approx \bigotimes_j |\beta_j\rangle}$$

Semiclassical approximation

$$\left|\psi\right\rangle \approx \bigotimes_j \left|\beta_j\right\rangle$$

In the position basis, this looks like $\left|\psi\right\rangle \approx \bigotimes_{\mathbf{x}} \left|\psi(\mathbf{x})\right\rangle$

Just a complex function of space, determining the particular coherent state at each point

Any expectation values are simple:

$$\left\langle \hat{\psi}^\dagger(\mathbf{x})\hat{\psi}^\dagger(\mathbf{x}')\hat{\psi}(\mathbf{x}')\hat{\psi}(\mathbf{x})\right\rangle = \psi^*(\mathbf{x})\psi^*(\mathbf{x}')\psi(\mathbf{x}')\psi(\mathbf{x})$$

$$\left\langle \hat{\psi}(\mathbf{x})\right\rangle = \psi(\mathbf{x})$$

"Mean-field"

(Mean-field approximation)

How would these coherent states evolve?

Semiclassical approximation

$$i\hbar\frac{\partial\hat{\psi}(\mathbf{x})}{\partial t}=\left(-\frac{\hbar^2}{2m}\nabla^2+V(\mathbf{x})+U\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x})\right)\hat{\psi}(\mathbf{x})$$

$$i\hbar\frac{\partial\langle\hat{\psi}(\mathbf{x})\rangle}{\partial t}=\left\langle\left(-\frac{\hbar^2}{2m}\nabla^2+V(\mathbf{x})+U\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x})\right)\hat{\psi}(\mathbf{x})\right\rangle$$

$$i\hbar\frac{\partial\psi(\mathbf{x})}{\partial t}=\left(-\frac{\hbar^2}{2m}\nabla^2+V(\mathbf{x})+U|\psi(\mathbf{x})|^2\right)\psi(\mathbf{x})$$

Gross-Pitaevskii equation (GPE), or
Non-linear Schrödinger equation

Why “semi-classical”?

$$i\hbar \frac{\partial \psi(\mathbf{x})}{\partial t} = \left(-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x}) + U|\psi(\mathbf{x})|^2 \right) \psi(\mathbf{x})$$

-Still looks “quantum”
Often called a “macroscopic wavefunction”

- We’ve ignored a lot of the quantum features

- Same approximation often made in quantum optics
  $\Rightarrow$ Replace a large, coherent mode with E($\mathbf{x}$,t)

Semiclassical variants

Careful examination of the Hamiltonian shows that the coherent states underlying our version of the mean-field approximation are not stable:

$$\hat{H} = \cdots + U \int d\mathbf{x}\, \hat{\psi}^{\dagger}(\mathbf{x})\hat{\psi}^{\dagger}(\mathbf{x})\hat{\psi}(\mathbf{x})\hat{\psi}(\mathbf{x})$$

conserves atom number, but not "coherent-ness"

There are variants of the derivation shown in this lecture that make a different approximation to the state of the field. The total field is still characterised by $\psi(\mathbf{x})$ (generally called an "order parameter")

$$i\hbar \frac{\partial \psi(\mathbf{x})}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x}) + U|\psi(\mathbf{x})|^2 \right) \psi(\mathbf{x})$$

What can we do with the GPE?

$$i\hbar \frac{\partial \psi(\mathbf{x})}{\partial t} = \left(-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x}) + U |\psi(\mathbf{x})|^2 \right) \psi(\mathbf{x})$$

Can’t be right all the time, but it’s surprisingly useful

- Spatial behaviour of BEC undergoing only linear processes
  - Evolution in any external potential (including time dependent)
  - Coupling between different internal states
  - Can be used to describe weak BEC excitations, BEC manipulation with optical or magnetic potentials, coupling between internal states, atom lasers, vortices, solitons, wave-guiding, feedback, …

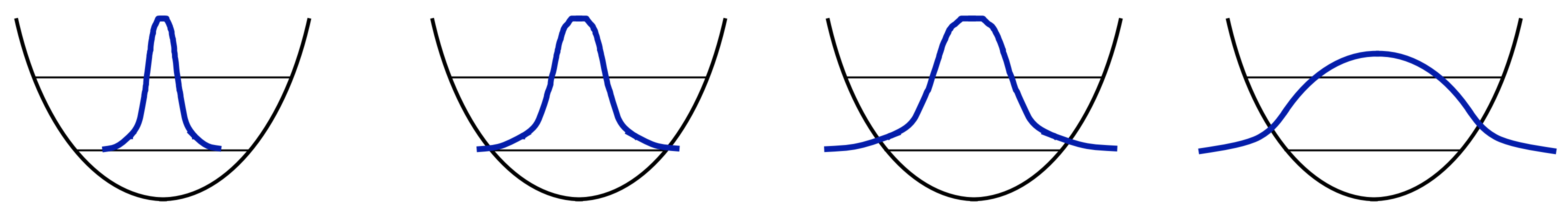
Trap ground state

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x}) + U\left|\psi(\mathbf{x})\right|^2\right)\psi(\mathbf{x}) = \mu\,\psi(\mathbf{x})$$

chemical potential

With U=0, the ground state looks like the ground state of a single particle in a trap. For harmonic traps, (and most are), this is a Gaussian.

$U < 0$ $\quad$ $U = 0$ $\quad$ $U > 0$ $\quad$ $U\left|\psi\right|^2 >> \hbar\omega$



Thomas-Fermi approximation

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x}) + U|\psi(\mathbf{x})|^2\right)\psi(\mathbf{x}) = \mu\,\psi(\mathbf{x})$$

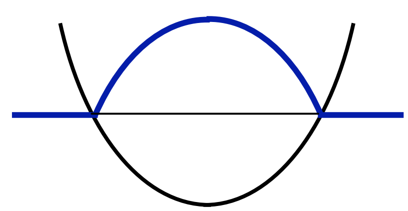
For sufficiently strong interactions… ignore kinetic energy

$$\psi(\mathbf{x}) \approx \sqrt{\frac{\mu - V(\mathbf{x})}{U}}$$

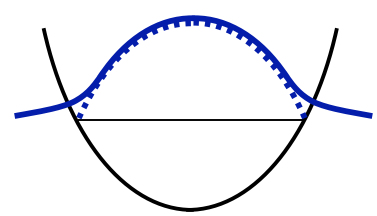
$$\int d\mathbf{x}\, |\psi(\mathbf{x})|^2 = \int d\mathbf{x}\, \frac{\mu - V(\mathbf{x})}{U} = N$$

chemical potential depends on the number of atoms

Wavefunction like an upside down potential



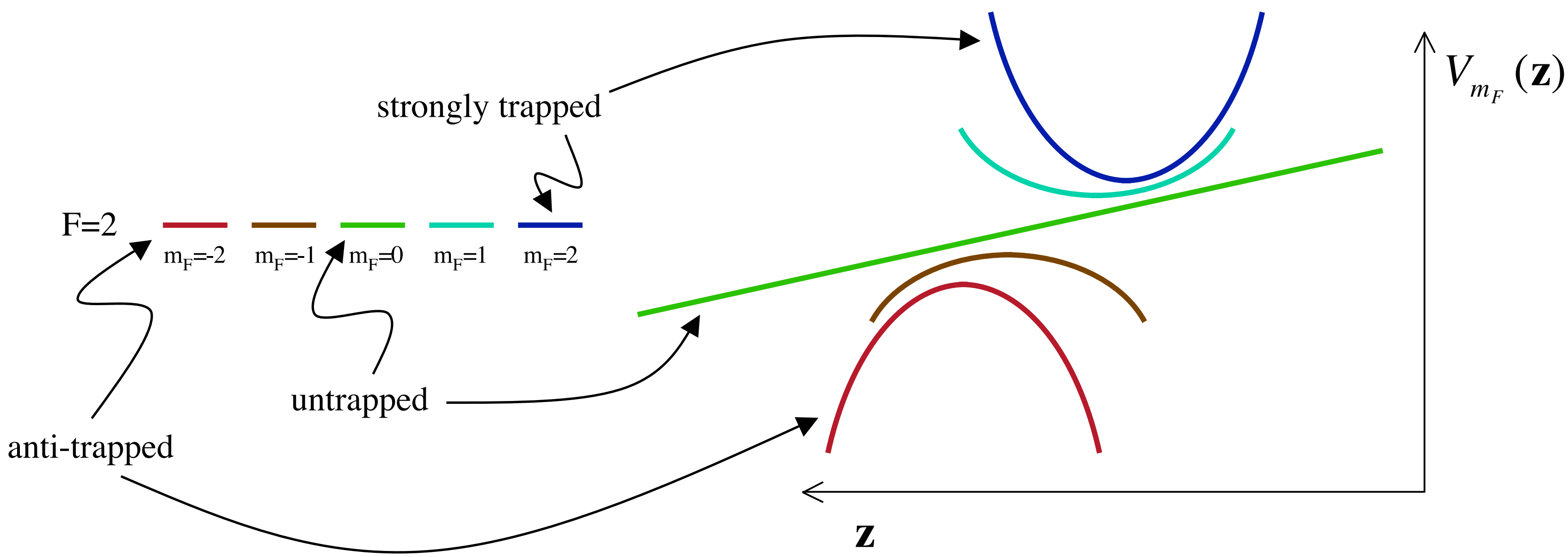
approximate



exact

The atom laser

Magnetic traps work on the magnetic moment of the atoms



Radio waves can cause transitions between neighbouring m_F states

How would we model this?

Modelling the atom laser (GPE)

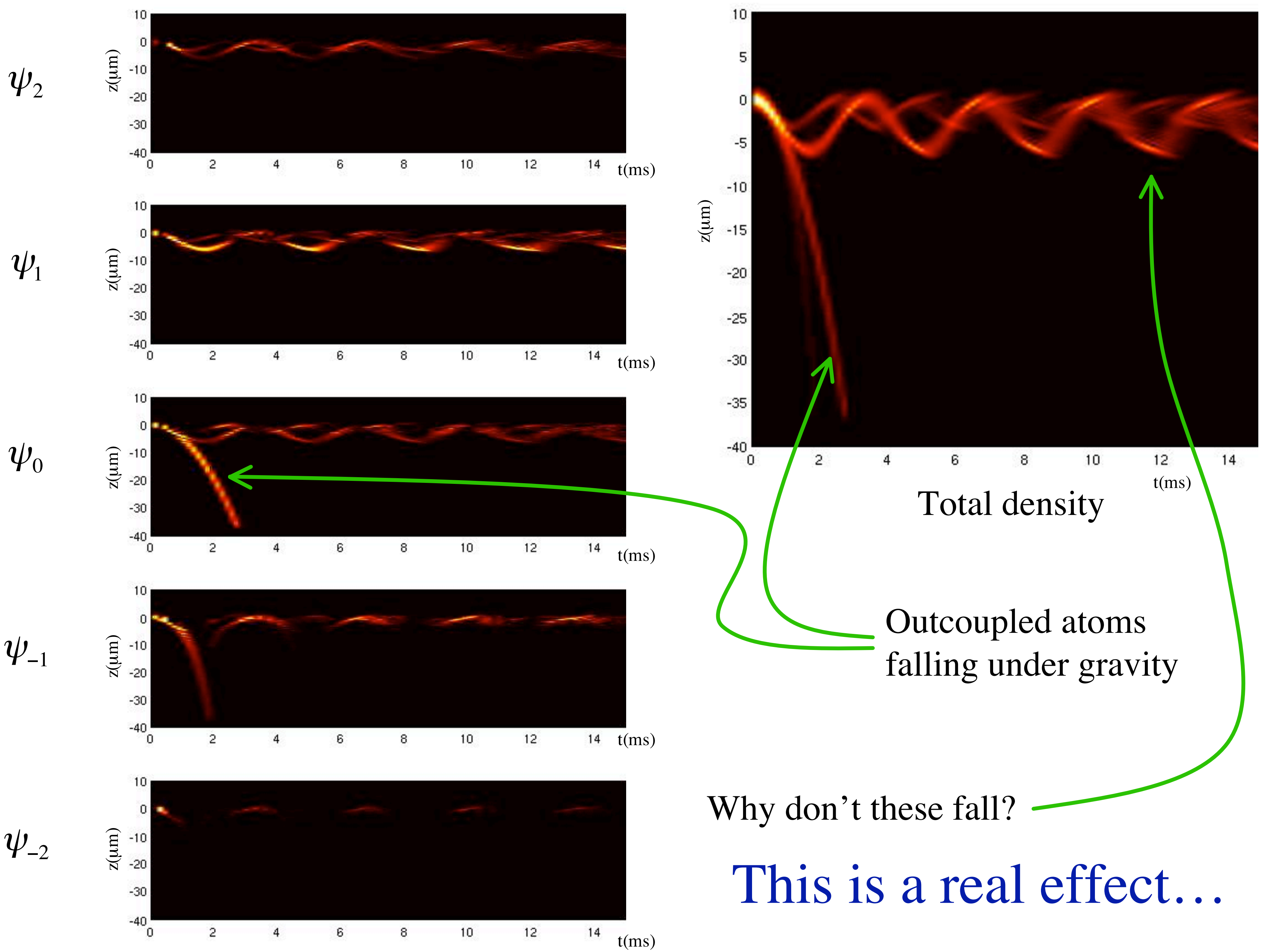
Each component of the field operator approximated by a classical field

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\psi_2(\mathbf{x})\\ \psi_1(\mathbf{x})\\ \psi_0(\mathbf{x})\\ \psi_{-1}(\mathbf{x})\\ \psi_{-2}(\mathbf{x})\end{pmatrix} = \left(-\frac{\hbar^2}{2m}\nabla^2 + U\rho(\mathbf{x}) + \begin{pmatrix} V_2(\mathbf{x}) & \hbar\Omega & 0 & 0 & 0 \\ \hbar\Omega & V_1(\mathbf{x}) & \sqrt{6}\hbar\Omega/2 & 0 & 0 \\ 0 & \sqrt{6}\hbar\Omega/2 & V_0(\mathbf{x}) & \sqrt{6}\hbar\Omega/2 & 0 \\ 0 & 0 & \sqrt{6}\hbar\Omega/2 & V_{-1}(\mathbf{x}) & \hbar\Omega \\ 0 & 0 & 0 & \hbar\Omega & V_{-2}(\mathbf{x}) \end{pmatrix}\right)\begin{pmatrix}\psi_2(\mathbf{x})\\ \psi_1(\mathbf{x})\\ \psi_0(\mathbf{x})\\ \psi_{-1}(\mathbf{x})\\ \psi_{-2}(\mathbf{x})\end{pmatrix}$$

$$\rho(\mathbf{x}) = |\psi_2(\mathbf{x})|^2 + |\psi_1(\mathbf{x})|^2 + |\psi_0(\mathbf{x})|^2 + |\psi_{-1}(\mathbf{x})|^2 + |\psi_{-2}(\mathbf{x})|^2$$

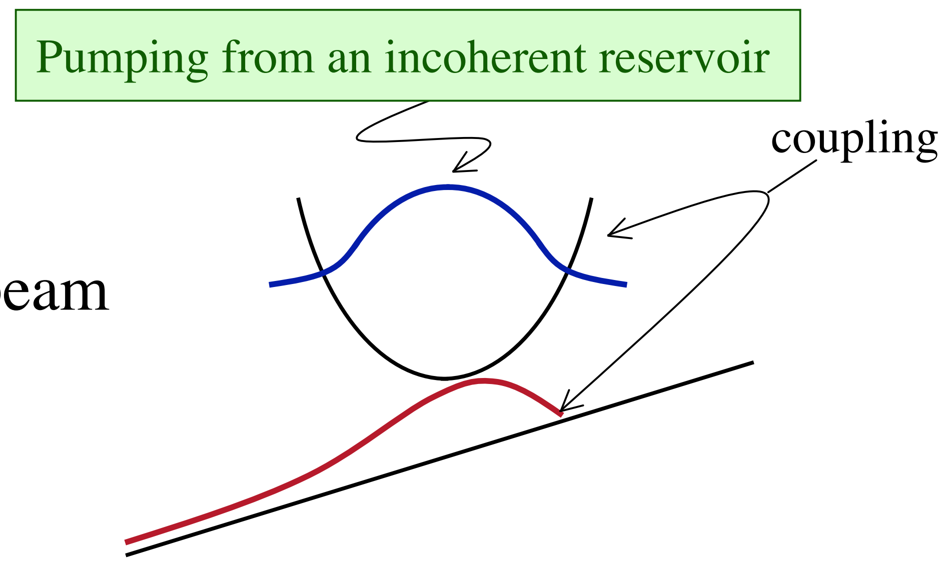
This is almost five lines of the GPE we derived earlier
- coupling between the different fields
- identical interactions between each component

Easy to put on a grid, and solve!



This is a real effect…

The pumped atom laser



Pump atoms into the BEC while coupling them out - continuous beam

How do we model this?

- A two-component quantum field, each with a different potential
- Hamiltonian with kinetic energy, potential energy, coupling between fields, and interactions between components
- Use semiclassical approximation $\rightarrow$ two classical fields
- Extra effects added phenomenologically

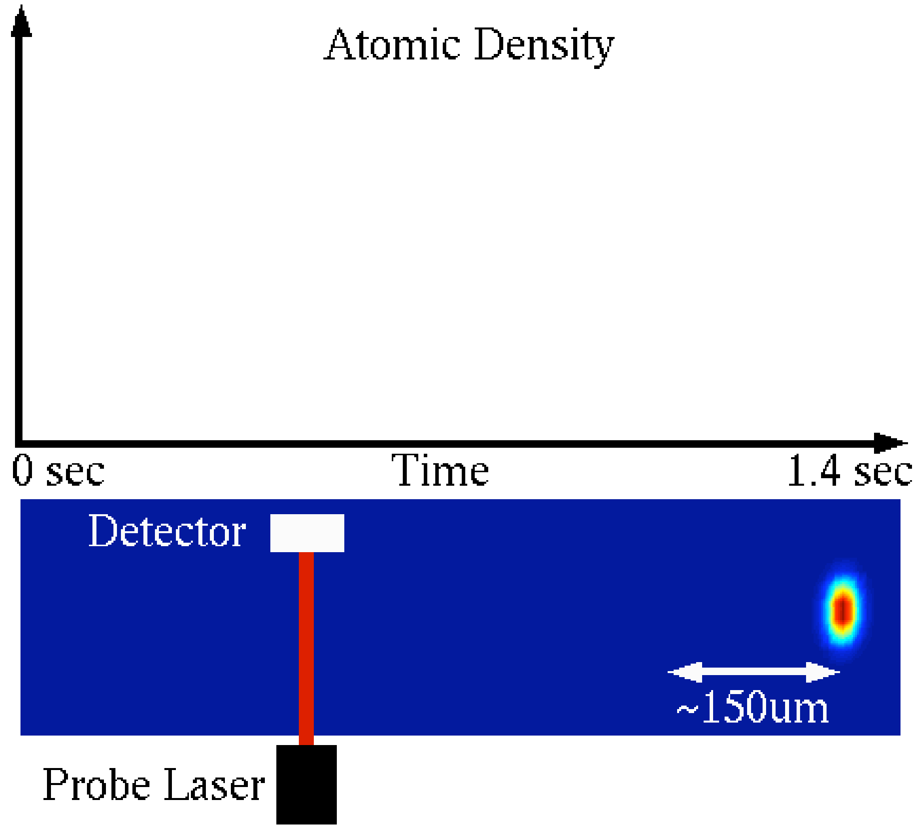
The pumped atom laser

$$i\hbar\frac{\partial}{\partial t}\psi_t(\mathbf{x}) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{trap}(\mathbf{x}) + U_{tt}|\psi_t|^2 + U_{tu}|\psi_u|^2 - i\hbar\gamma_t^{(1)} - i\hbar\gamma_{tt}^{(2)}|\psi_t|^2 - i\hbar\gamma_{tu}^{(2)}|\psi_u|^2 + i\kappa\rho\right)\psi_t + \hbar\Omega e^{i\mathbf{k.x}}\psi_u$$

$$i\hbar\frac{\partial}{\partial t}\psi_u(\mathbf{x}) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{gravity}(\mathbf{x}) + U_{uu}|\psi_u|^2 + U_{tu}|\psi_t|^2 - i\hbar\gamma_u^{(1)} - i\hbar\gamma_{uu}^{(2)}|\psi_u|^2 - i\hbar\gamma_{tu}^{(2)}|\psi_t|^2\right)\psi_u + \hbar\Omega e^{-i\mathbf{k.x}}\psi_t$$

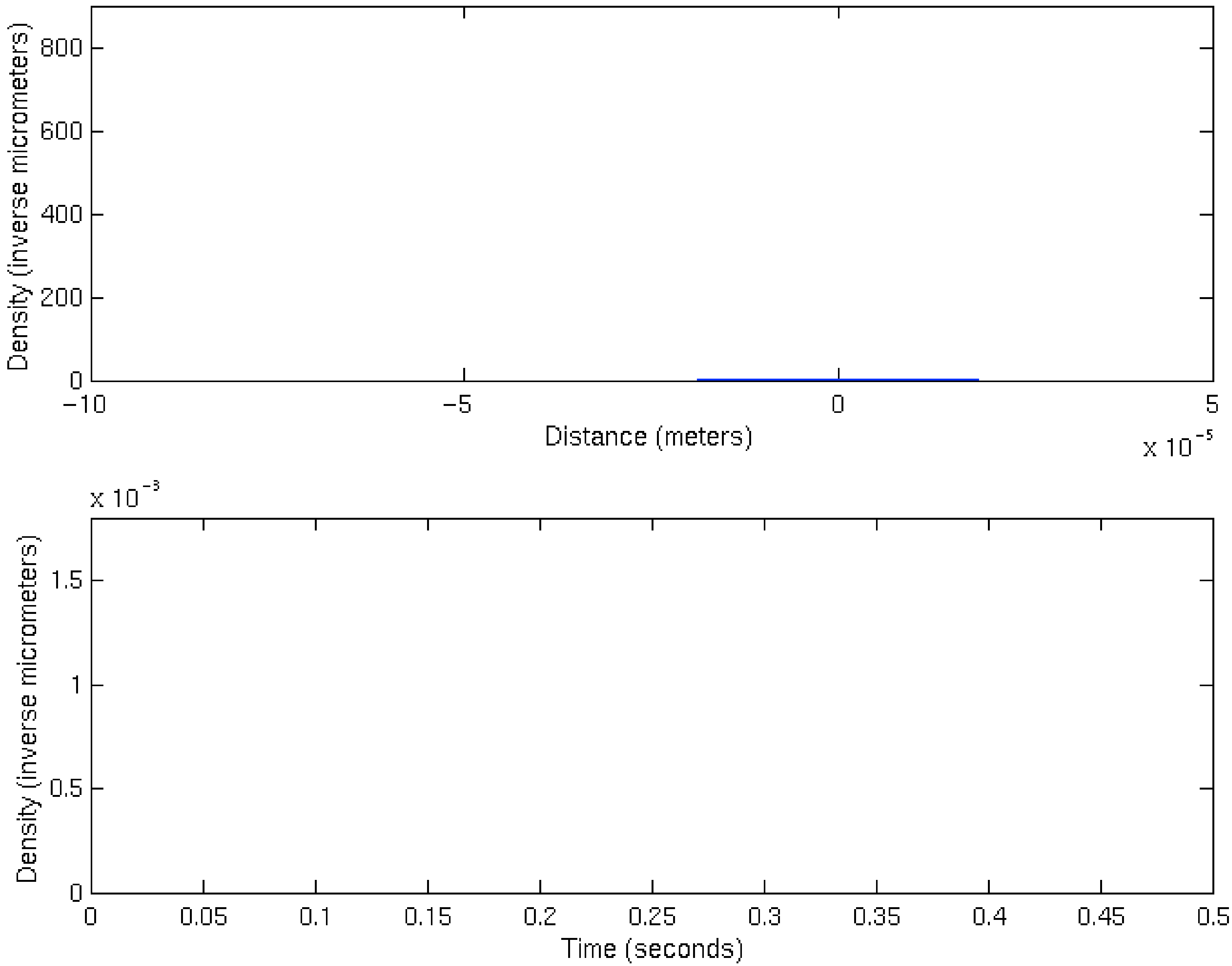
$$\frac{\partial}{\partial t}\rho(\mathbf{x}) = r - \gamma_{res}\rho - \kappa|\psi_t|^2\rho + \lambda\nabla^2\rho$$

- BEC-BEC, beam-beam and BEC-beam interactions
- BEC-BEC, beam-beam and BEC-beam inelastic scattering
- Gravity, trapping potentials, background gas losses
- Momentum kick during output coupling
- Spatial coupling and pumping

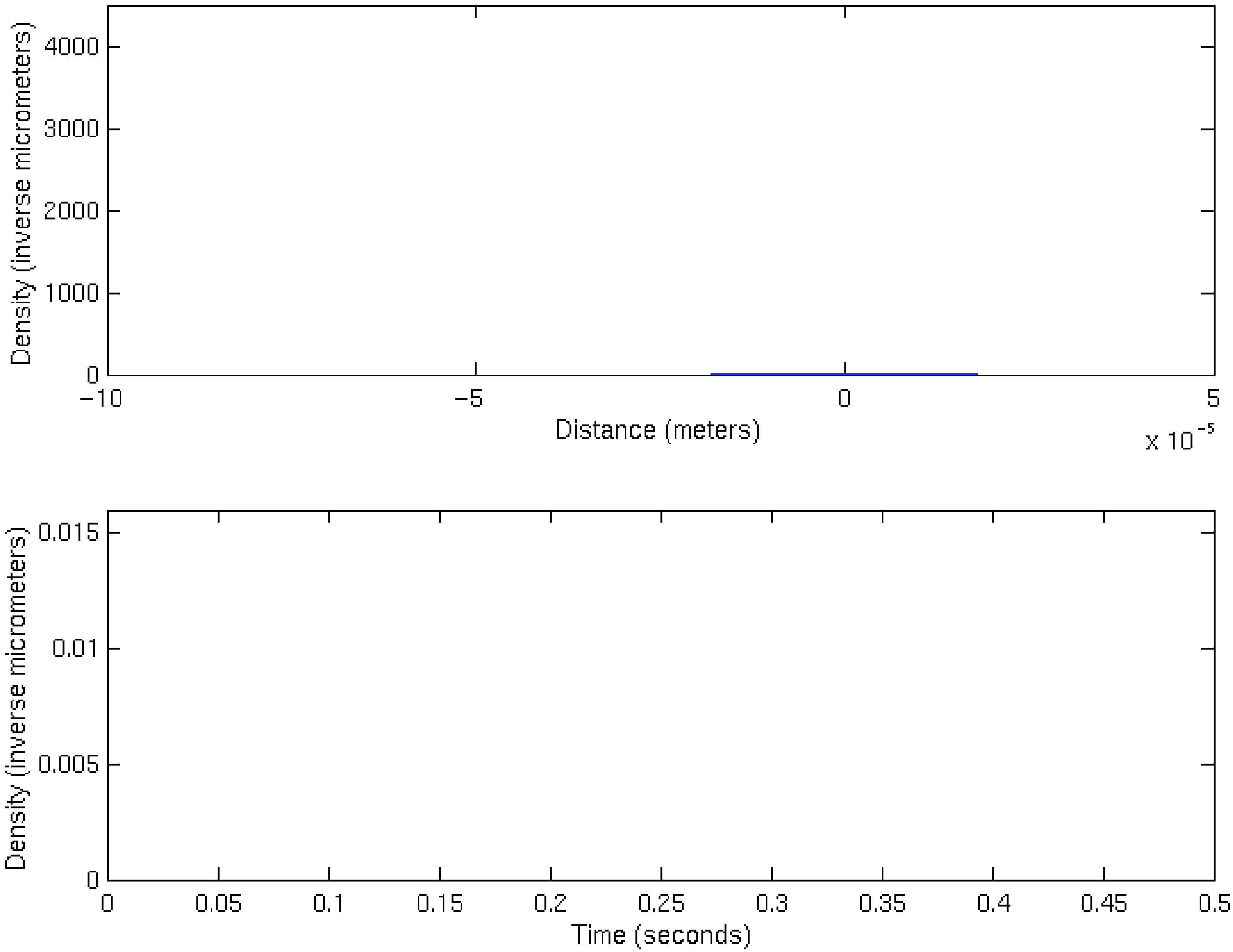


Mode selectivity?

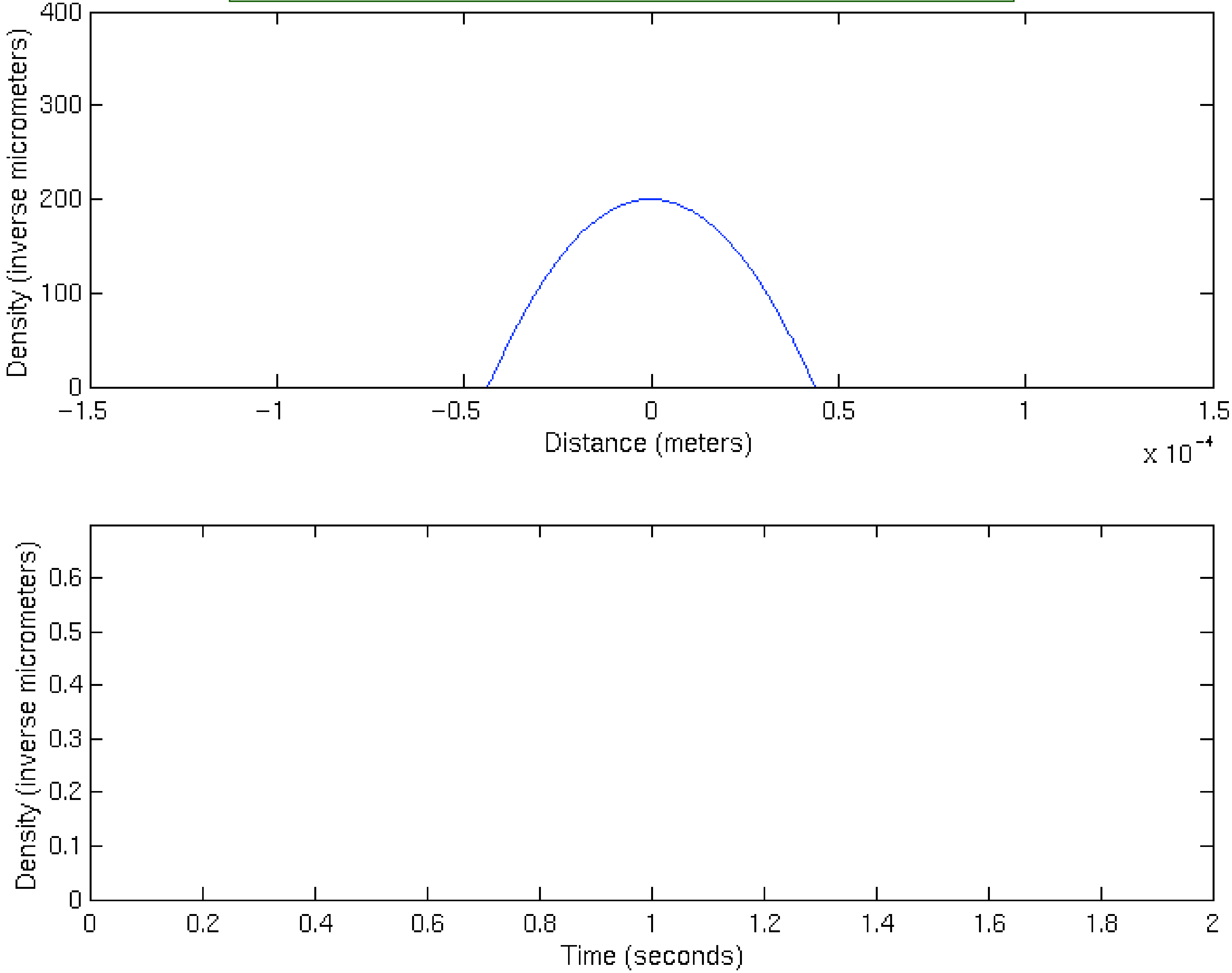
No interactions, weak pumping



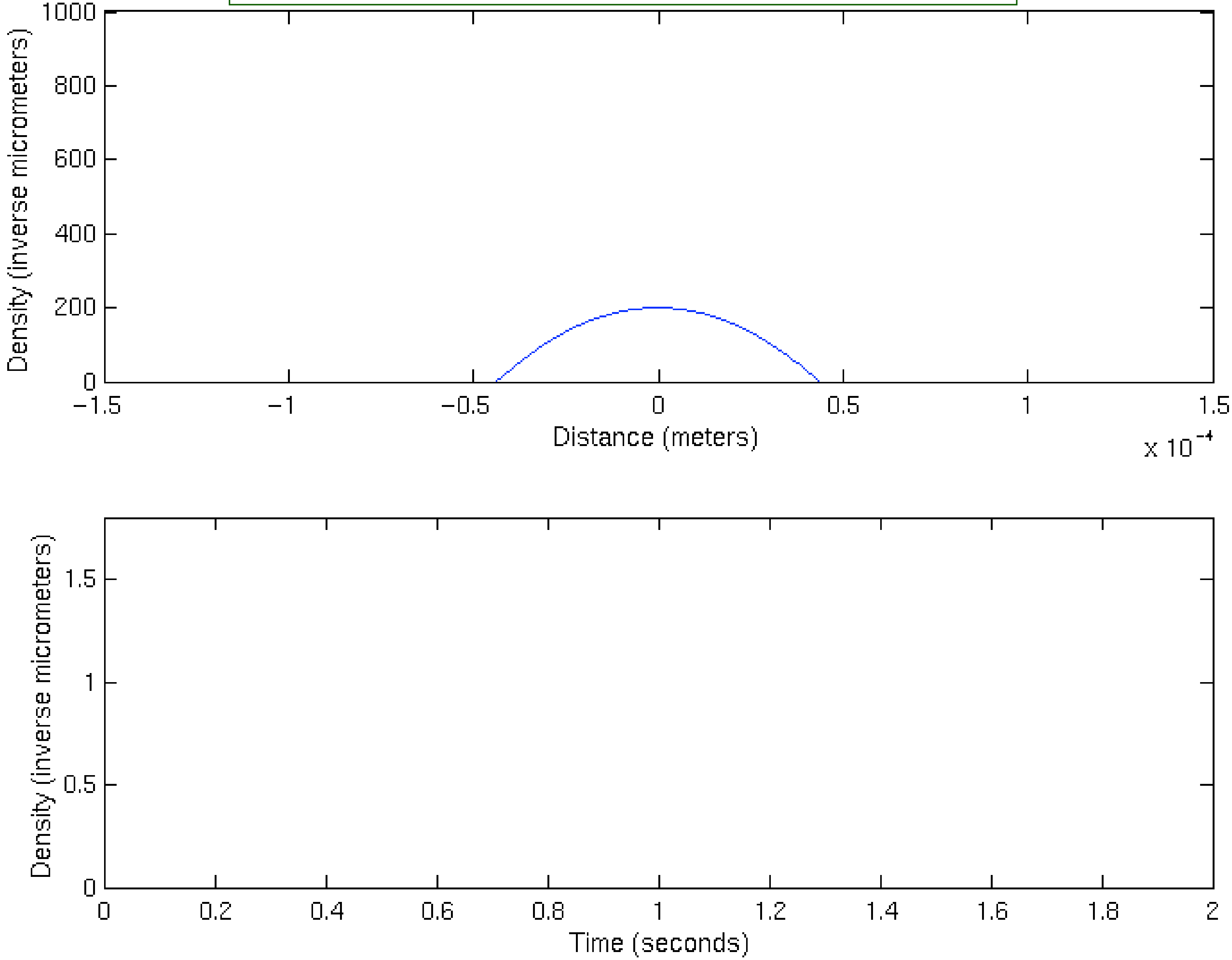
No interactions, strong pumping



Moderate interactions, weak pumping

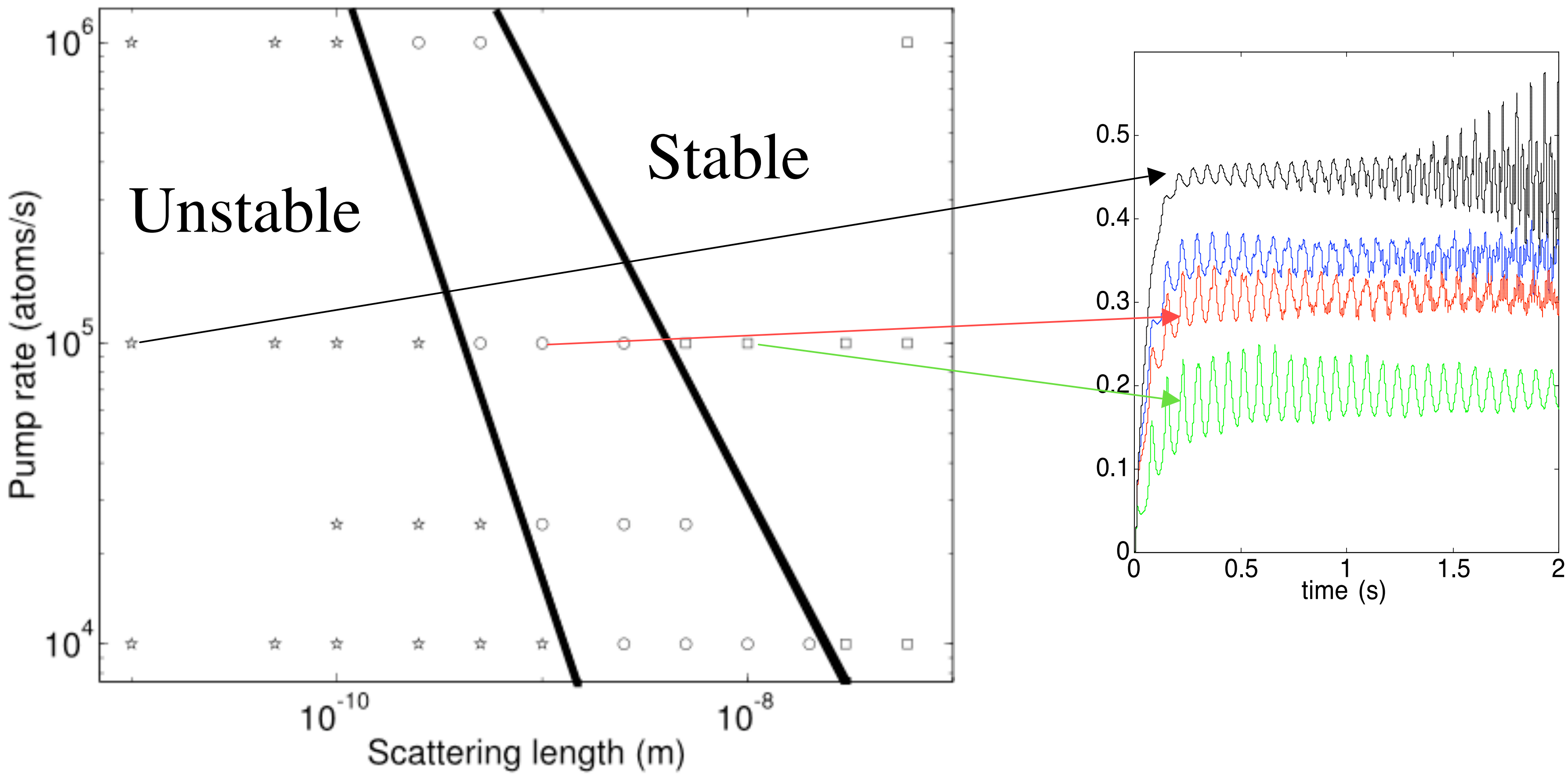


Moderate interactions, strong pumping

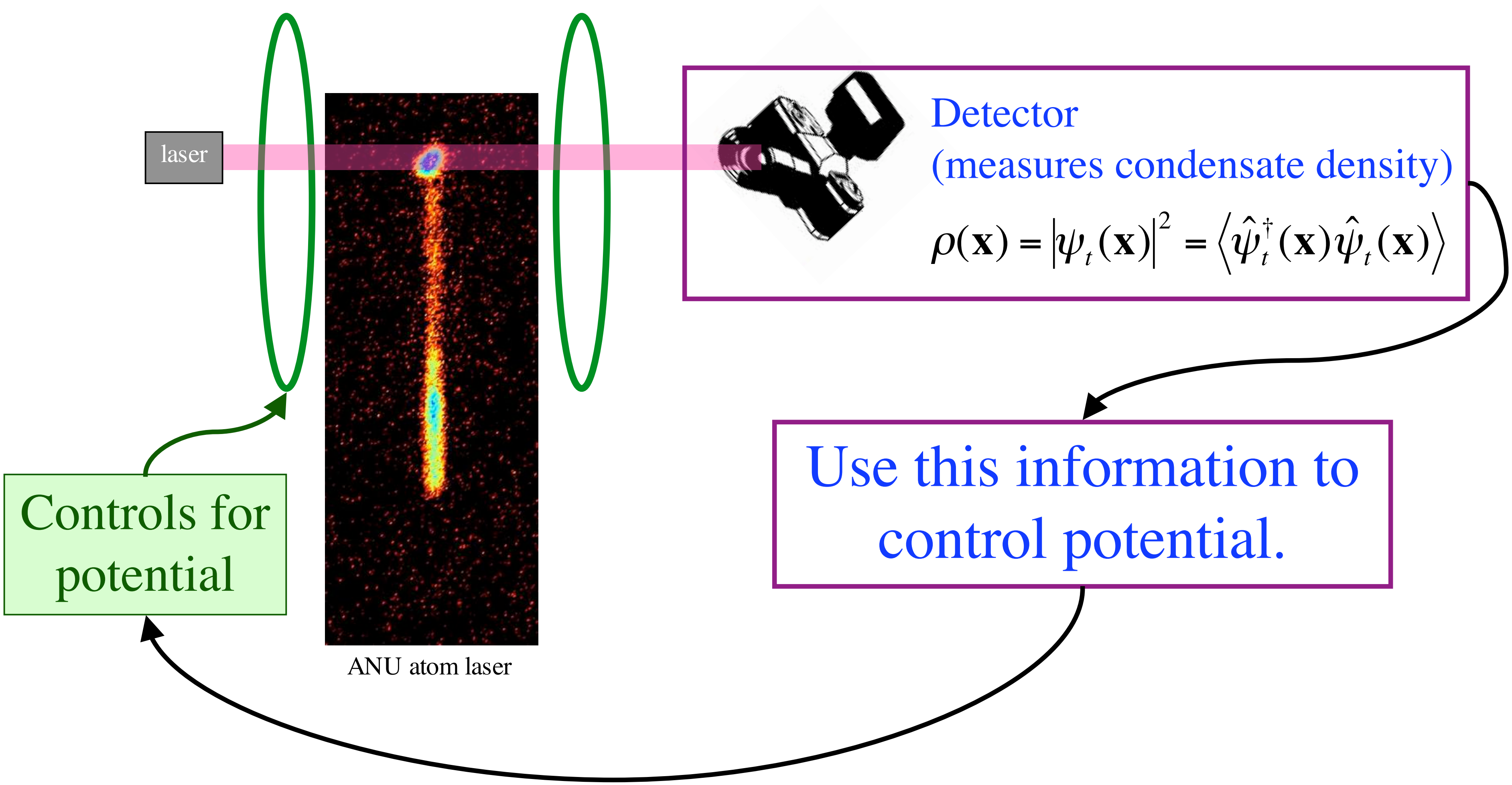


Stability of a pumped atom laser

Stability depends on scattering length and pumping rate



Feedback


laser
Detector
(measures condensate density)
$\rho(\mathbf{x}) = |\psi_t(\mathbf{x})|^2 = \langle \hat{\psi}_t^\dagger(\mathbf{x})\hat{\psi}_t(\mathbf{x})\rangle$
Use this information to control potential.
Controls for potential
ANU atom laser

Exercises

$$\hat{H} = \int d\mathbf{x}\, \hat{\psi}^\dagger(\mathbf{x})\left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x},t)\right)\hat{\psi}(\mathbf{x}) + \frac{U(t)}{2}\int d\mathbf{x}\, \hat{\psi}^\dagger(\mathbf{x})\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x})\hat{\psi}(\mathbf{x})$$

$$V(\mathbf{x},t) = V_{trap}(\mathbf{x}) + V_{feedback}(\mathbf{x},t) \qquad U(t) = U_0 + U_{feedback}(t)$$

1. Show that this Hamiltonian leads to the following GPE:

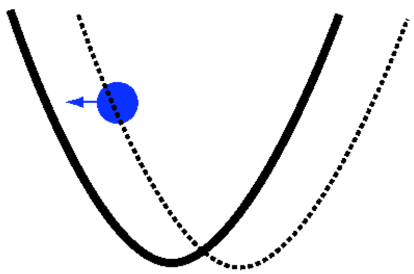
$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{x}) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{trap}(\mathbf{x}) + V_{feedback}(\mathbf{x},t) + \left(U_0 + U_{feedback}(t)\right)|\psi|^2\right)\psi$$

2. Show the following results

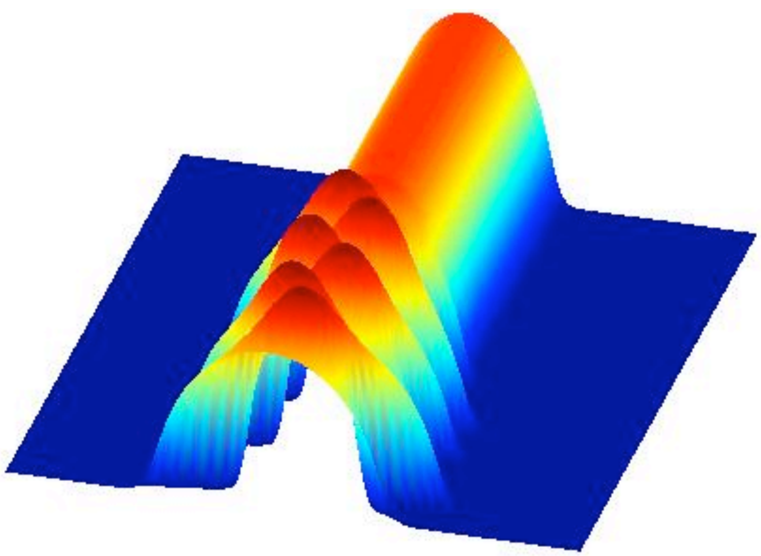
$$E_0 \equiv \left\langle \hat{H} \right\rangle\Big|_{\substack{V_{feedback}(\mathbf{x},t)=0 \\ U_{feedback}(t)=0}} = \int\left(-\frac{\hbar^2}{2m}\psi^*\nabla^2\psi + V_{trap}|\psi|^2 + \frac{U_0}{2}|\psi|^4\right)d\mathbf{x}$$

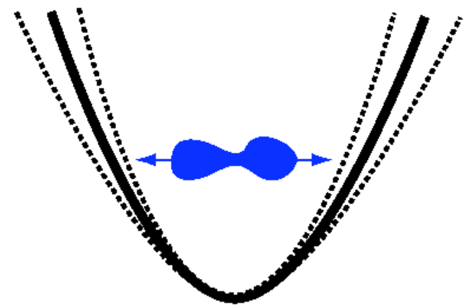
$$\frac{\partial E_0}{\partial t} = -\frac{\partial}{\partial t}\left(\int V_{feedback}(\mathbf{x},t)|\psi|^2\, d\mathbf{x} + \int U_{feedback}(t)|\psi|^4\, d\mathbf{x}\right)$$

Feedback

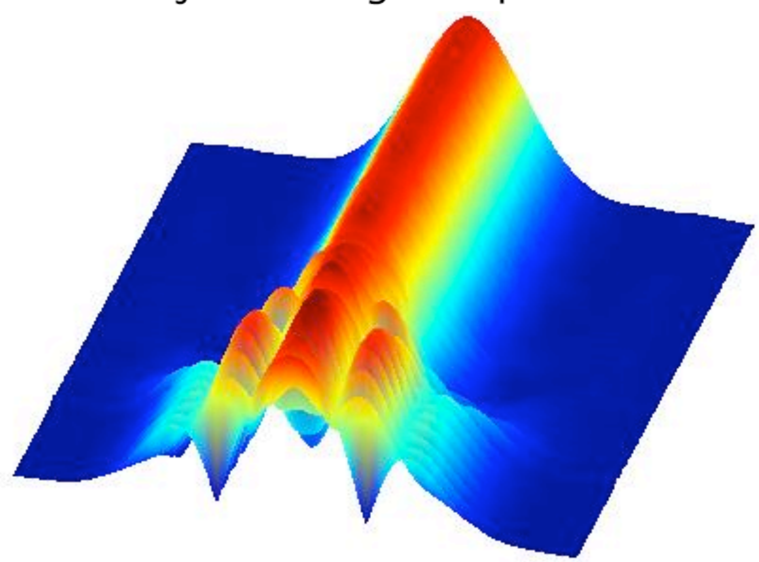


Sloshing?
Offset potential





Breathing?
Adjust strength of potential



Tomorrow

Methods of doing calculations where the quantum nature of the fields is important

- Atom lasers of the future