Phase and Interference in Bose Condensates

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This Talk

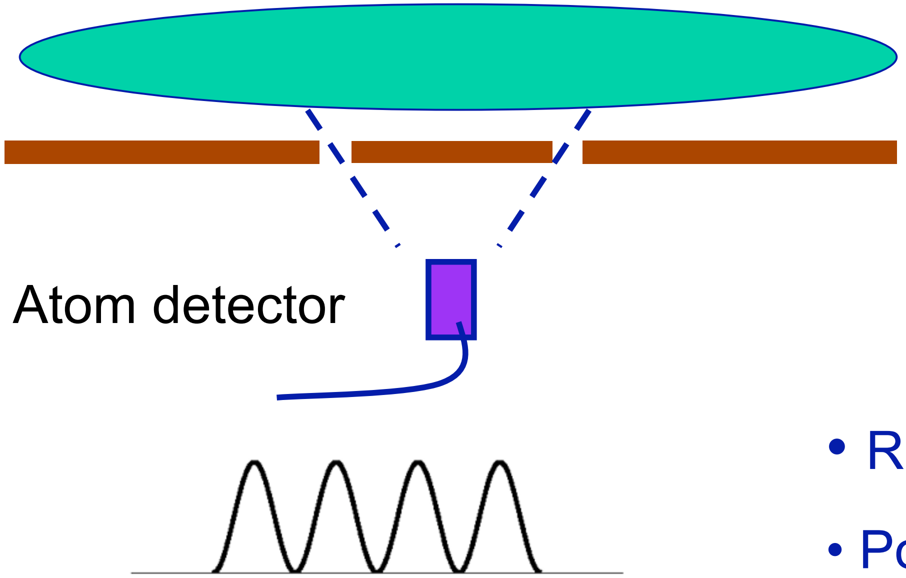
- Importance of phase
    - single condensate
    - interference of two condensates
- Establishing relative phase between condensates
    - role of quantum measurement
    - atom number conservation
- Quantum phase operator
- Single quantum trajectory
    - stochastic Schrodinger equation
- Phase standard, phase transfer, phase locking

Single condensate

Defining characteristic: **Phase coherence**

Condensate differs fundamentally from a cloud of cold atoms

Issue is quantum interference



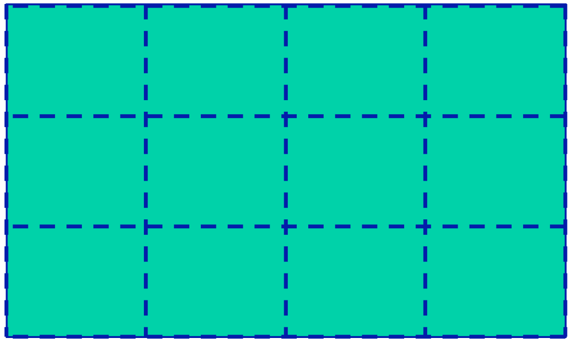
Experiment of
Hansch group, Munich

Coherence means

- Relative phase predictable
- Position of fringe known in advance

Simple demonstration of significance of phase

Homogeneous BEC, N atoms



Divide into k equal boxes

Cannot treat as a collection of boxes, each with a condensate of N/k atoms

Introduce a_i annihilation operator for box i ; $\langle a_i^\dagger a_i \rangle = N/k$

annihilation operator for WHOLE condensate $A_0 = 1/\sqrt{k} \sum_i a_i$

$$\begin{aligned} N &= \langle A_0^\dagger A_0 \rangle = \frac{1}{k} \sum_i \langle a_i^\dagger a_i \rangle + \frac{1}{k} \sum_{i \neq j} \langle a_i^\dagger a_j \rangle \\ &= \frac{N}{k} + \frac{1}{k} \sum_{i \neq j} \langle a_i^\dagger a_j \rangle . \end{aligned}$$

$$N = \langle A_0^\dagger A_0 \rangle = \frac{1}{k}\sum_i \langle a_i^\dagger a_i \rangle + \frac{1}{k}\sum_{i\neq j}\langle a_i^\dagger a_j \rangle \quad = \frac{N}{k} + \underbrace{\frac{1}{k}\sum_{i\neq j}\langle a_i^\dagger a_j \rangle}_{\text{Must contribute significantly}}.$$

i.e. must have phase correlations

$$\langle a_i^\dagger a_j \rangle = N/k$$

(can show phase difference between any two points is zero)

or, in terms of field operator

$$\hat{\psi}(x) = \sum_k a_k \phi_k(x)$$

$\phi_k(x)$ is a mode function

phase correlation is expressed

$$\langle \psi^\dagger(x)\psi(y) \rangle = \rho_0$$

For all x, y

Leads to a general definition of a BEC …..(but that’s another story)

A first approach to describing phase properties

Broken symmetry   Order parameter (condensate wavefunction)

Ground state wavefunction

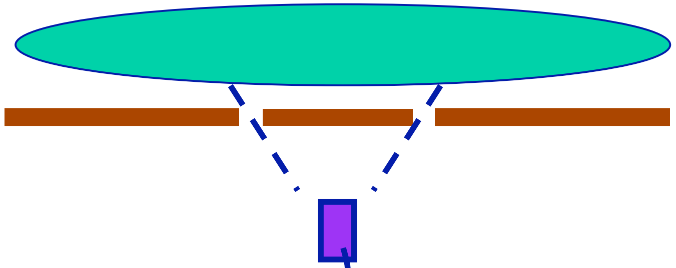
$$\Psi(\mathbf{r}) = |\Psi(\mathbf{r})|e^{i\theta}$$

θ independent of x

Make the **assumption:**

a particular θ is chosen spontaneously

(analogy to ferromagnet)



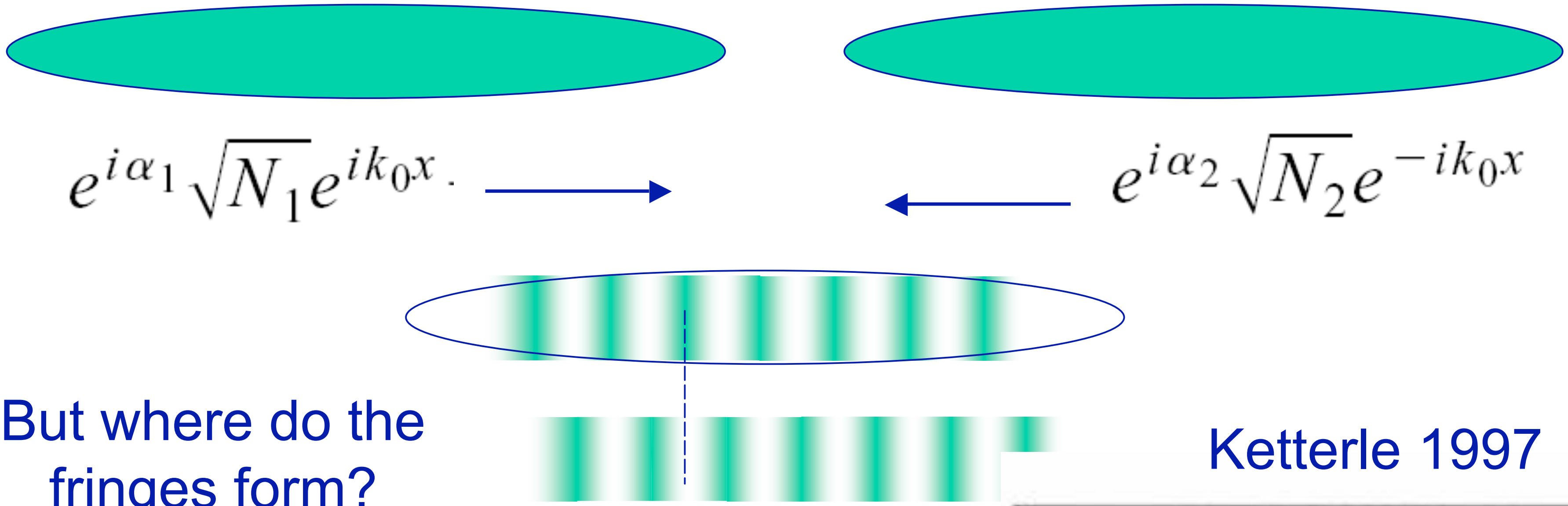
Now, the description of *single condensate interference*

→ similar to (classical) Young's experiment for light

Two condensates

What is the ***Relative phase of two condensates ?***

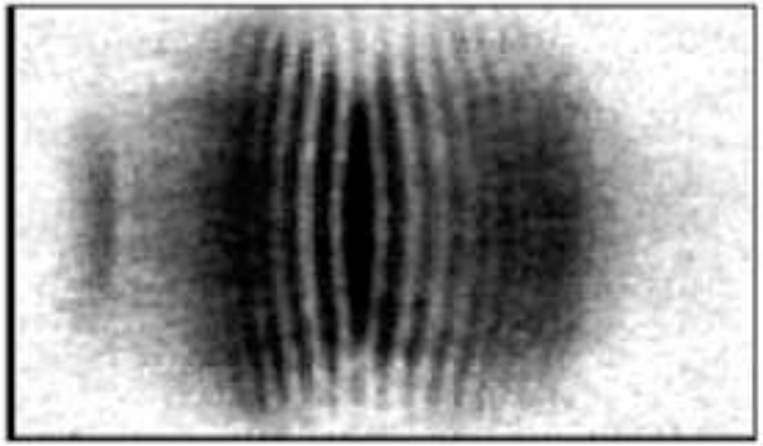
Broken symmetry treatment



$e^{i\alpha_1}\sqrt{N_1}e^{ik_0x}.$ $\longrightarrow$ $\longleftarrow$ $e^{i\alpha_2}\sqrt{N_2}e^{-ik_0x}$

But where do the fringes form?

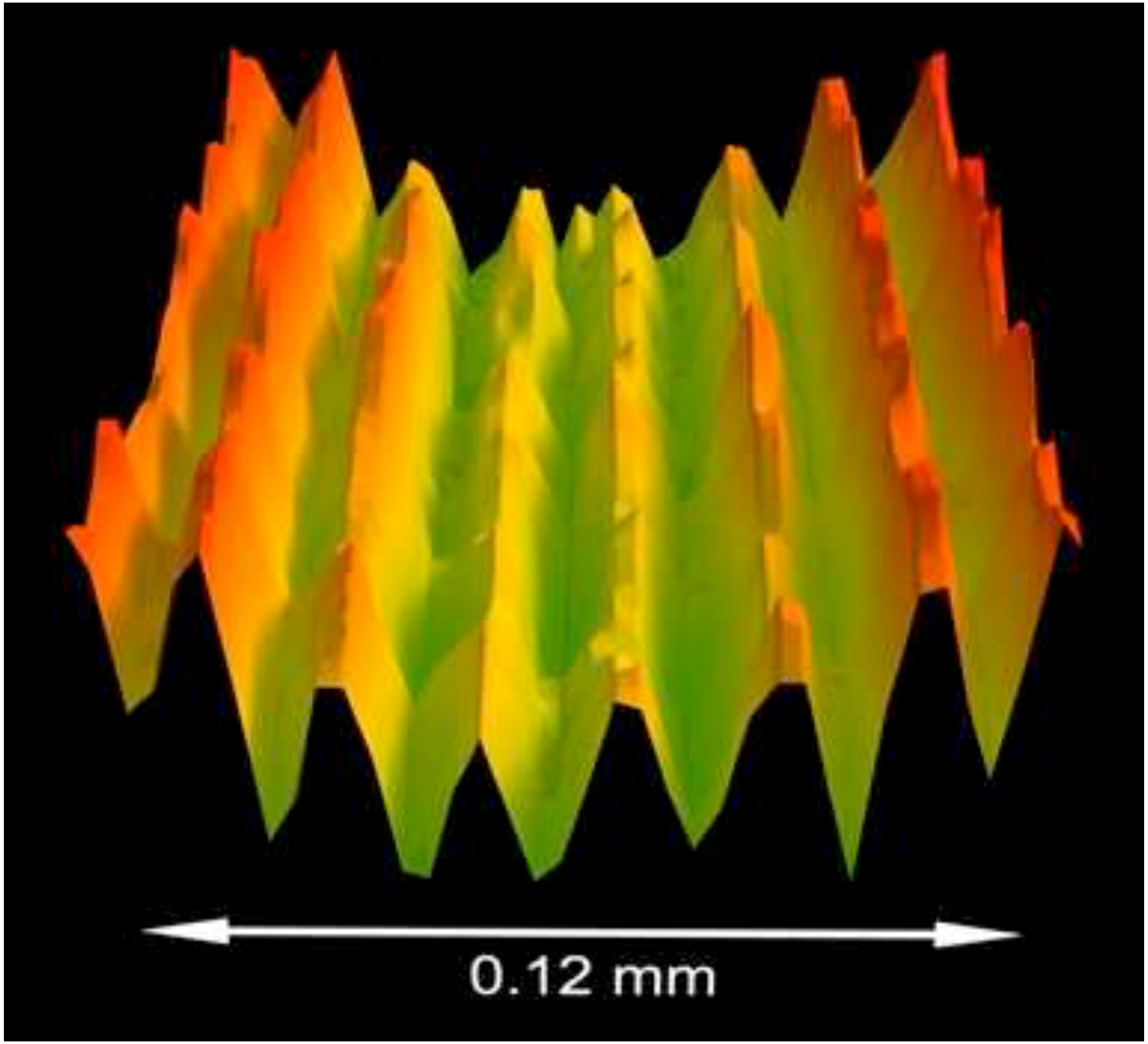
Ketterle 1997



In this treatment, depends on the relative phase

$$\alpha_1 - \alpha_2$$

Andrews, …, Ketterle, Science 275, 637 (1997)



Problems with the Broken symmetry approach

1. Unlike ferromagnetic case, there is no small field to align the phase
2. More importantly, it is not in accord with the requirement of

   particle number conservation

In *second-quantized* theory of non-relativistic QM, all observables commute with the number operator $\hat{N}$

Leads to a super selection rule – number of particles is conserved

Let's try to do the calculation more carefully.

Consider a two-mode condensate state, e.g. modes are

e^{ik_0x} and e^{-ik_0x}

Assume initial condensate state is $|\Psi\rangle = |N_1\rangle|N_2\rangle$

The field operator is $\hat{\psi}(x) = \frac{1}{\sqrt{L}}\left(a_1 e^{ikx} + a_1 e^{-ikx}\right)$

The operator for one particle density is $\hat{\psi}^\dagger(x)\hat{\psi}(x)$

Let's get a number for it. The usual way is …

$$\langle\hat{\psi}^\dagger(x)\hat{\psi}(x)\rangle = \langle N_1|\langle N_2|\hat{\psi}^\dagger(x)\hat{\psi}(x)|N_1\rangle|N_2\rangle$$

$$= (N_1 + N_2)/L$$

This is uniform (***no interference***)

because we have calculated an **ensemble** average.

We need to calculate a **single realisation** of the experiment

There are a number of different approaches …

But first let's understand the

Quantum Phase Operator

Barnett & Pegg, J. Mod Opt, **36**, 7 (1989) (and a series of papers)

Consider single harmonic oscillator mode, frequency ω

Begin by recalling states of ***precisely defined phase*** exist

$$|\theta\rangle = \lim_{s \to \infty} (s+1)^{-1/2} \sum_{n=0}^{s} \exp(in\theta)|n\rangle$$

Possesses some important properties we'd expect for phase

(i) The associated field (e.g. EM) goes to zero at $\theta - \omega t = m\pi$

(ii) Furthermore $|\theta\rangle \rightarrow |\theta - \omega t\rangle$

Can define (s+1) orthonormal phase states that span the space

$$|\theta_m\rangle = \exp[i\hat{N} m 2\pi/(s+1)]|\theta_0\rangle, \quad m=0,1,\ldots s \ .$$

$$\theta_m = \theta_0 + \frac{2\pi m}{(s+1)}$$

Hermitian Phase operator $\hat{\phi}_\theta \equiv \sum_{m=0}^{s} \theta_m |\theta_m\rangle\langle\theta_m|$

$$\left[\hat{N}, \hat{\phi}_\theta\right] \neq 0$$

Noncommuting !

So cannot know **both** phase and number precisely

e.g. Consider Fock state $|N>$

$$\langle\hat{\phi}_\theta\rangle = \theta_0 + \pi,$$

$$\Delta\phi_\theta^2 = \pi^2/3$$

Same as uniform random classical distribution

$$\bar{\phi} = \frac{1}{2\pi}\int_{\theta_0}^{\theta_0+2\pi} \phi \, \mathrm{d}\phi = \theta_0 + \pi,$$

$$\mathrm{Var}\,\phi = \frac{1}{2\pi}\int_{\theta_0}^{\theta_0+2\pi} (\phi - \bar{\phi})^2 \, \mathrm{d}\phi = \pi^2/3$$

e.g. *Consider Coherent state* $a|\alpha\rangle = \alpha|\alpha\rangle$

$$\alpha = r\exp(\mathrm{i}\xi)$$

$$|\alpha\rangle = \exp(-r^2/2)\sum_n \frac{r^n}{\sqrt{(n!)}}\exp(\mathrm{i}n\xi)|n\rangle$$

$$P(\theta) = |\langle\theta|\alpha\rangle|^2 \approx \frac{2\pi}{s+1}\left(\frac{4r^2}{2\pi}\right)^{1/2}\exp[-2r^2(\xi-\theta)^2] \qquad (r^2 \gg 1)$$

$$\langle\hat{\phi}_\theta\rangle = \xi. \qquad \Delta\phi_\theta^2 = \frac{1}{4\bar{n}}$$

$\bar{n} = r^2$ is mean particle number

$\Delta N^2 = \bar{n}$ (Poisson distribution)

$\Delta N \Delta\phi_\theta = \frac{1}{2}$ *i.e.* Heisenberg Uncertainty

Emphasize: **Phase Operator**

allows calculation of **mean** and **width** of phase distribution

We need **relative phase** $\hat{\phi}_1 - \hat{\phi}_2$

If we ignore the width of the distribution,
simple proxy for the relative phase between two modes

$$|\Psi\rangle = |\theta_1\rangle|\theta_2\rangle$$

$$\boxed{Arg\{\langle\Psi|a_1^\dagger a_2|\Psi\rangle\} = (\theta_2 - \theta_1)}$$

Now, return to

problem of ***Relative phase of two condensates***

Number of atoms in condensate is fixed (in any one run)

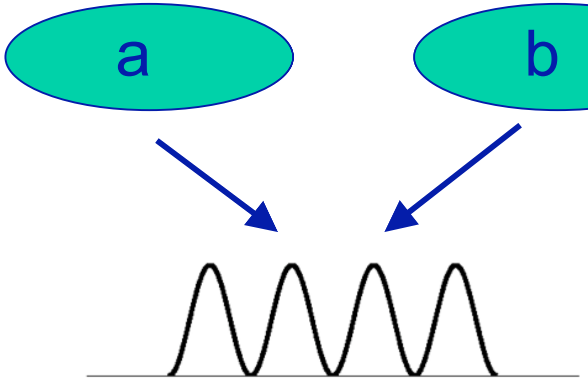
implies phase totally undefined

Creation of Phase by measurement

There a number of approaches; Simplest, for our purposes

(Dunningham & Burnett PRL, **82**,3279 (1999))

Basic idea: Condensate in two modes [Ignore (collisional) interactions]



Operators for condensate in mode *a*

$$a^\dagger\ ,\ a$$

Operators for condensate in mode *b*

$$b^\dagger\ ,\ b$$

Each initially in number state $|N>$, *i.e* initial state is $|N\rangle_a|N\rangle_b$

Allow modes to overlap, detect an atom in such a way **we cannot know** which mode it came from i.e. apply, $\frac{1}{\sqrt{2}}(a + b\mathrm{e}^{i\theta(x)})$

detect atom in either mode *a* or mode *b*

$\theta(x)$ Is a phase that depends on where the atom is detected

Apply **detection**

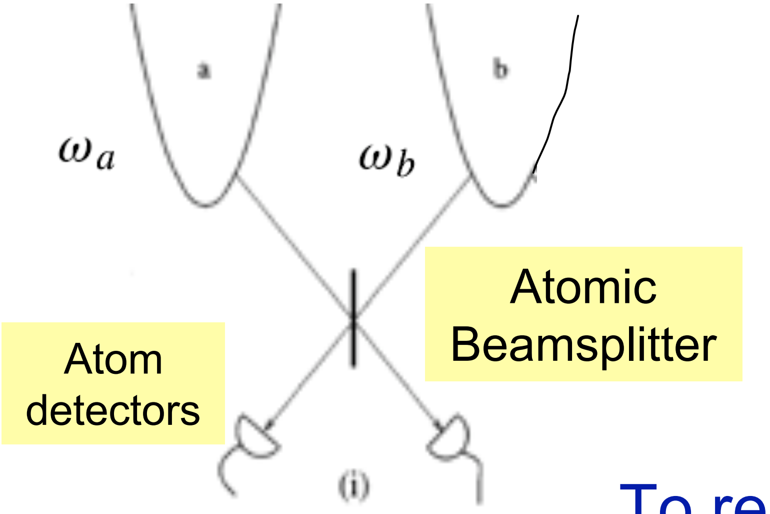
$$\frac{1}{\sqrt{2}}(a + b\mathrm{e}^{i\theta(x)})|N\rangle_a|N\rangle_b = \underbrace{\frac{1}{\sqrt{2}}\left(|N-1\rangle_a|N\rangle_b + \mathrm{e}^{i\theta(x)}\,|N\rangle_a|N-1\rangle_b\right)}_{\text{state after measurement}}$$

Important Points

1. Atom numbers accounted for exactly
2. The two modes are now entangled
3. $\langle a^\dagger b\rangle$ is now nonvanishing (coherence is now established)

More Formal approach

Dunningham & Burnett PRL, 82,3279 (1999)



Field operators at output ports

(in frame rotating at ω_a)

$$C_1 = \sqrt{\kappa/2}\,(a + ibe^{-i\Omega t})$$

Phase shift from reflection

$$C_2 = \sqrt{\kappa/2}\,(ia + be^{-i\Omega t}) \qquad \Omega = \omega_b - \omega_a$$

κ is the rate of detection of atoms

To realise a single quantum trajectory, use a Stochastic Schrodinger equation $(\hbar = 1)$

$$d|\psi_c\rangle = \Bigg\{dN_1\left(\frac{C_1}{\sqrt{Q_c^1(t)}} - 1\right) + \underset{\text{Noise}}{dN_2}\underbrace{\left(\frac{C_2}{\sqrt{Q_c^2(t)}} - 1\right)}_{\text{Quantum Jump operator}}$$

$$- dt\Bigg[\underbrace{iH_0 + \frac{\kappa}{2}(a^\dagger a + b^\dagger b}_{\text{Effective Hamiltonian (includes loss due to detection)}} - \langle a^\dagger a\rangle - \langle b^\dagger b\rangle)\Bigg]\Bigg\}|\psi_c\rangle$$

$$Q_c^i(t) \equiv \langle\psi_c(t)|C_i^\dagger C_i|\psi_c(t)\rangle$$

There is a rigorous basis for this

Comes from Master equation (which gives evolution of whole ensemble)

e.g.

$$\dot{\rho}=-i\omega[\hat{a}^{\dagger}\hat{a}+\hat{b}^{\dagger}\hat{b},\rho]+\kappa(2\hat{a}\rho\hat{a}^{\dagger}-\hat{a}^{\dagger}\hat{a}\rho-\rho\hat{a}^{\dagger}\hat{a})$$
$$+\kappa(2\hat{b}\rho\hat{b}^{\dagger}-\hat{b}^{\dagger}\hat{b}\rho-\rho\hat{b}^{\dagger}\hat{b}),$$

We are extracting one realisation, in such a way that an ensemble of realisations will match the predictions of the Master equation

See,

Carmichael, ***An Open Systems Approach to Quantum Optics*** Springer Lecture Notes in Physics, (1993)

Gardiner & Zoller ***Quantum Noise*** (2nd Edition) Springer

Implementation of quantum trajectory method

1. Calculate probability atom is detected at port i in time interval Δt

$$P_i(t, \Delta t) = \Delta t \langle \psi_c(t) | C_i^\dagger C_i | \psi_c(t) \rangle$$

2. Generate uniform random numbers r_i $0 < r_i < 1$

3. If $r_i < P_i$ a detection is made in port i, and system jumps to

$$|\psi_c(t)\rangle \to \frac{C_i|\psi_c(t)\rangle}{\sqrt{\langle\psi_c(t)|C_i^\dagger C_i|\psi_c(t)\rangle}}$$

4. If $r_1 > P_1$ and $r_2 > P_2$ no detection in the interval.
Propagate system as

$$|\psi_c(t)\rangle \to \exp[-iH_{\text{eff}}\Delta t]|\psi_c(t)\rangle$$

5. Repeat until reach final time

For convenience, consider temporal fringes, instead of spatial fringes

(allows us to use trap eigenstates instead of momentum eigenstates)

Relative phase between *a* and *b* $\phi_{ab}(t) = \arg\{\langle\psi_c(t)|a^\dagger b|\psi_c(t)\rangle\}$

Can relate to ***difference in number of atoms*** detected in each port

Detection rate per unit time

$$D(t)/\Delta t = \langle\psi_c(t)\,|C_1^\dagger C_1 - C_2^\dagger C_2|\psi_c(t)\rangle$$

$$= i\kappa(\langle a^\dagger b\rangle e^{i\Omega t} - \langle b^\dagger a\rangle e^{-i\Omega t})$$

Use

$$\langle a^\dagger b\rangle = |\langle a^\dagger b\rangle| e^{i\phi_{ab}(t)}$$

$$D(t)/\Delta t = -2\kappa\sin[\Omega t + \phi_{ab}(t)]$$

Results:

Numerical simulation $\omega_a = \omega_b/4 = 40\kappa$

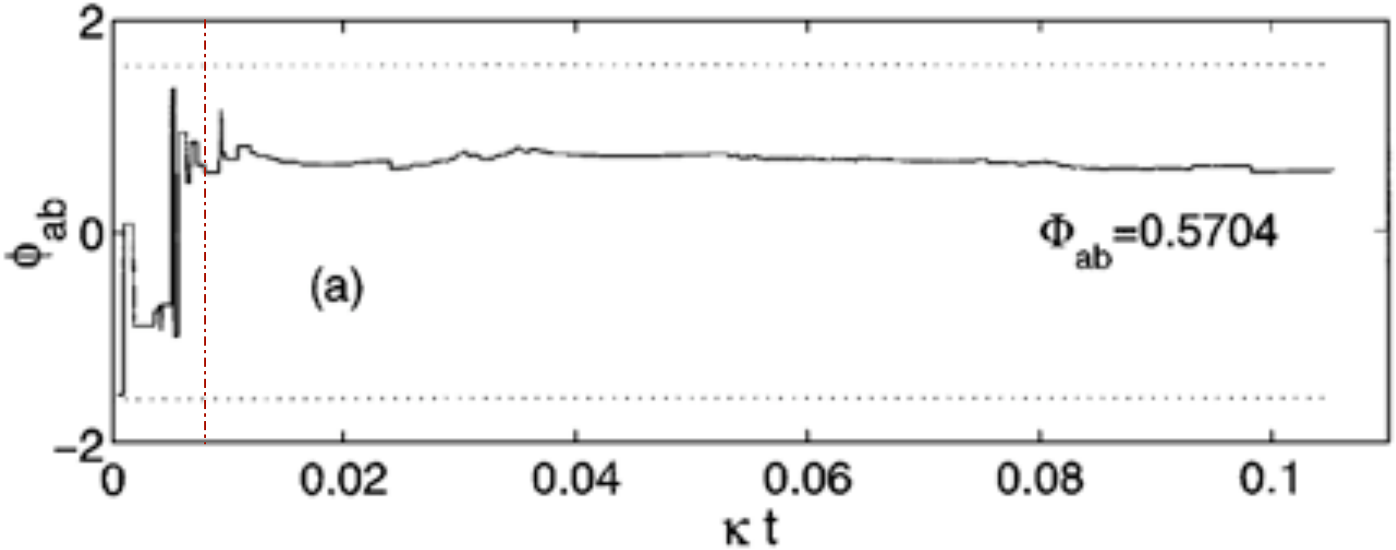
(Removed deterministic phase Ωt)

Initial state

$$|N\rangle_a|N\rangle_b$$

N=1000

Phase initially undefined



At the end of the detection process (say 10% atoms lost) modes *a* and *b* are **entangled**

$$\boxed{|\psi_c\rangle = \sum_{i=N-l}^{N} c_i|2N - l - i\rangle_a|i\rangle_b}$$

l atoms detected

$(l < N)$

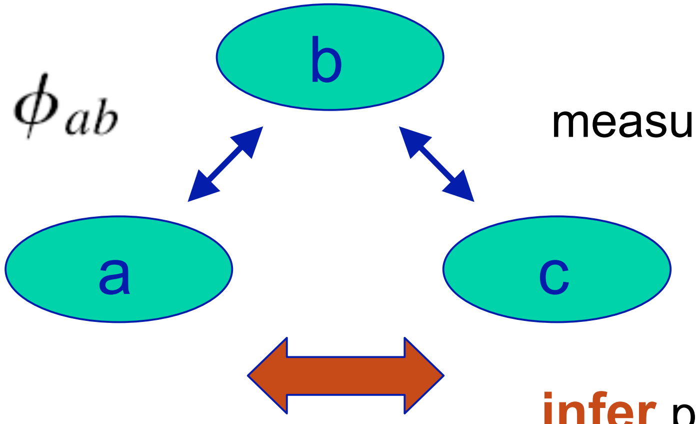
Measurement has created the phase

Phase Standard

All phase is relative. However, can we seek a

reference condensate

with which other condensates can be compared ?



measure phase ϕ_{ab}

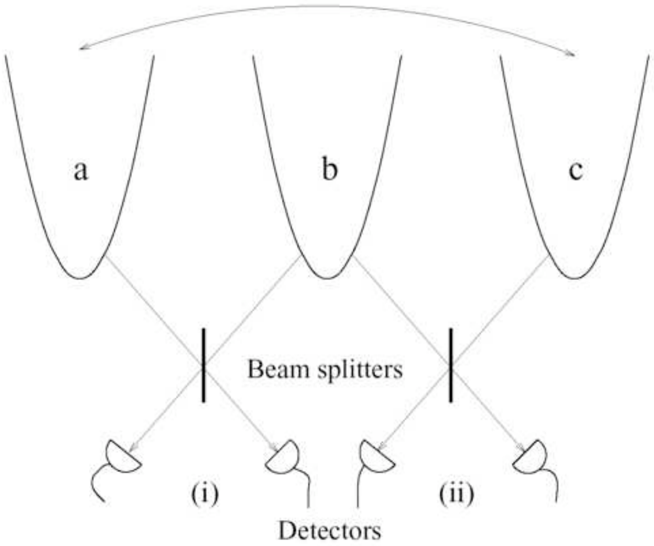
measure phase ϕ_{cb}

infer phase ϕ_{ac}

Extend previous calculation

Initial state

$$|N\rangle_a |N\rangle_b |N\rangle_c$$



1. Use 10% of atoms in *a* and *b* to establish Φ_{ab}

a and *b* now entangled

$$|\psi_c\rangle = \left(\sum_{i=N-l}^{N} c_i |2N - l - i\rangle_a |i\rangle_b \right) |N\rangle_c$$

2. Now detect on second beamsplitter

$$|\psi_c\rangle = \sum_{j=N-m}^{N} \sum_{i=2N-m-j}^{N} d_{i,j} |3N - i - j - m\rangle_a |j\rangle_b |i\rangle_c$$

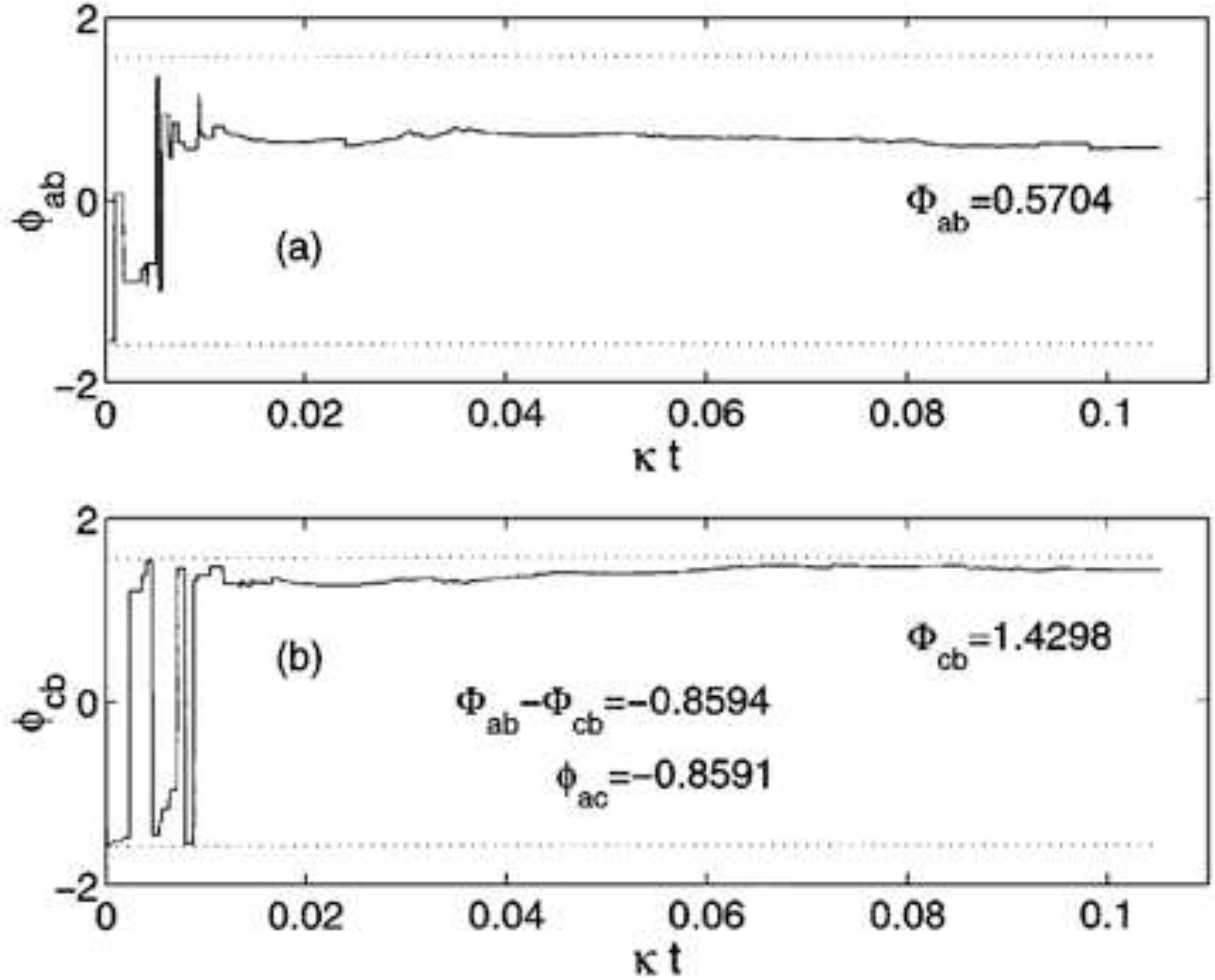
Important points:

1. entanglement between *a* and *b* maintained
2. And now *a,b* and *c* are entangled
3. Must keep *a* isolated from other systems (e.g. environment)

Results:

Numerical simulation

$$\omega_a = \omega_c = \omega_b/4 = 40\kappa$$



$$\phi_{ac} = \arg[\langle\psi_c|a^\dagger c|\psi_c\rangle]$$

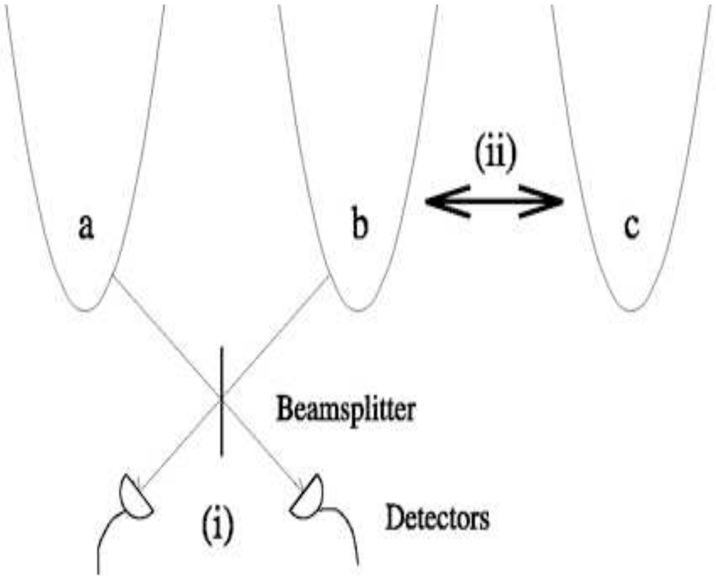
and check that

$$\phi_{ac} = \Phi_{ab} - \Phi_{cb}$$

Phase Transfer

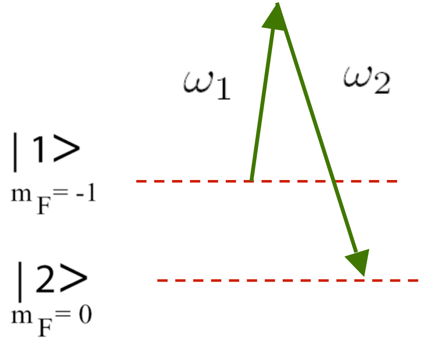
[Dunningham & Burnett J Phys B, 33,3807 (2000)]

$$|N\rangle_a|N\rangle_b|0\rangle_c$$

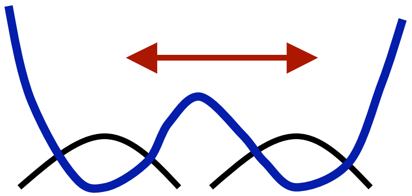


Coherent coupling *b* - *c*

e.g. Raman to different hyperfine state



e.g. Josephson tunnelling to nearby well



Hamiltonian for transfer (neglecting condensate collisional interaction)

$$H = \omega_b b^\dagger b + \omega_c c^\dagger c + \Gamma \left(b c^\dagger \, \mathrm{e}^{-\mathrm{i}\omega_{\mathrm{dr}} t} + b^\dagger c \, \mathrm{e}^{\mathrm{i}\omega_{\mathrm{dr}} t} \right)$$

Coupling strength

Driving frequency

In interaction picture $$H_{\mathrm{I}} = \Gamma \left(c^{\dagger} b \exp(\mathrm{i}\delta t) + b^{\dagger} c \exp(-\mathrm{i}\delta t) \right)$$

$$\delta = \omega_c - \omega_b - \omega_{\mathrm{dr}}$$

Evolution operator $$U_{bc} = \exp(-\mathrm{i} H_{\mathrm{I}} t)$$ *Deterministic evolution*

After transfer $$|\psi_c\rangle = \sum_{i,j} e_{i,j} |2N - l - i - j\rangle_a |j\rangle_b |i\rangle_c$$

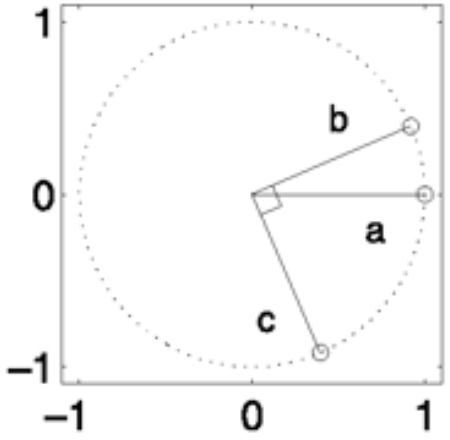
$\{e_{i,j}\}$ are coefficients determined by the numerical simulation

Procedure:

1. measure phase (i.e. establish a as reference)
2. transfer b -> c
3. Calculate ϕ_{ca} and ϕ_{ba} Hence determine ϕ_{bc}

Results:

First consider $\boxed{\delta = 0}$



Find from simulations,
that *b* always leads *c* by $\pi/2$.

Get same result for two (Josephson) coupled modes

Analytic calculation

$$\phi_{bc}(t) = \arg\left\{\langle 0|\langle N|\exp\left[\mathrm{i}\Gamma t\left(c^\dagger b + b^\dagger c\right)\right] c^\dagger b \exp\left[-\mathrm{i}\Gamma t\left(c^\dagger b + b^\dagger c\right)\right]|N\rangle|0\rangle\right\}$$

After some algebra

$$\phi_{bc}(t) = \arg\left\{\tfrac{1}{2}\mathrm{i}N\sin(2\Gamma t)\right\}$$

i.e. $|\phi_{bc}|$ has the value $\pi/2$ for all times except $t = \dfrac{m\pi}{2\Gamma}$

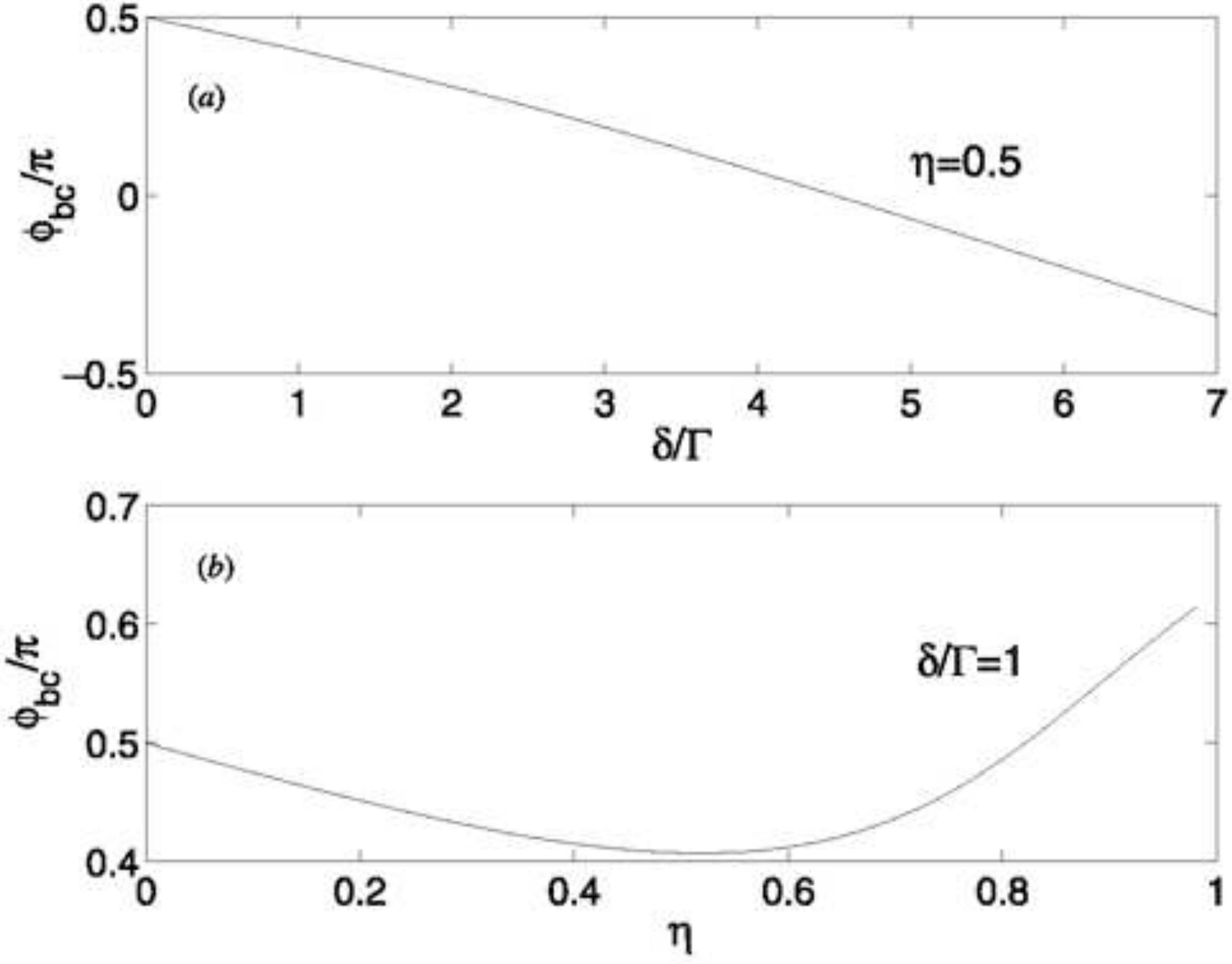
(population all in one mode)

Results:

$$\boxed{\delta \neq 0}$$

Numerical simulation

Phase can be controlled by choice of δ
(with careful choice of transfer time)



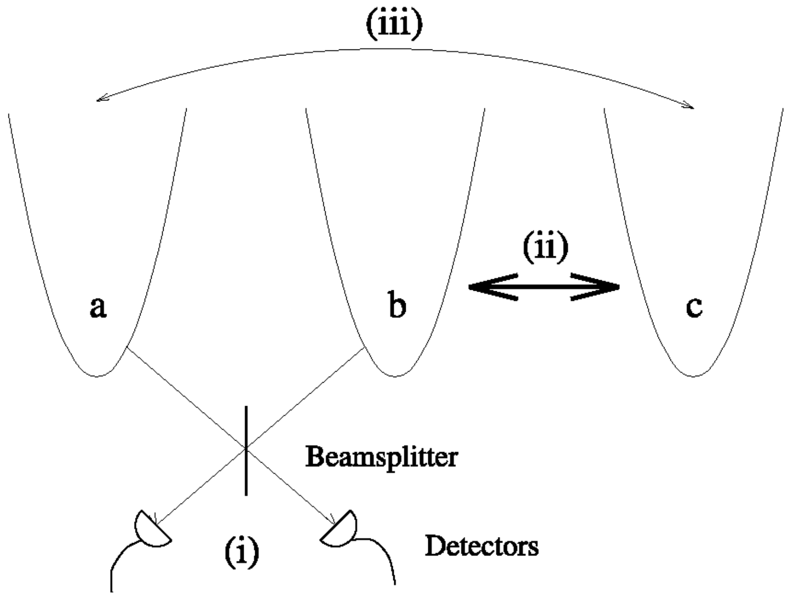
Scaled time

$$\eta = t/t_0$$

$$t_0 = \pi/2\Gamma$$

Phase Locking (i)

(with measurement)



Same setup, but now do
measurement (a-b)
and
coherent transfer (b-c)
simultaneously

Initial state

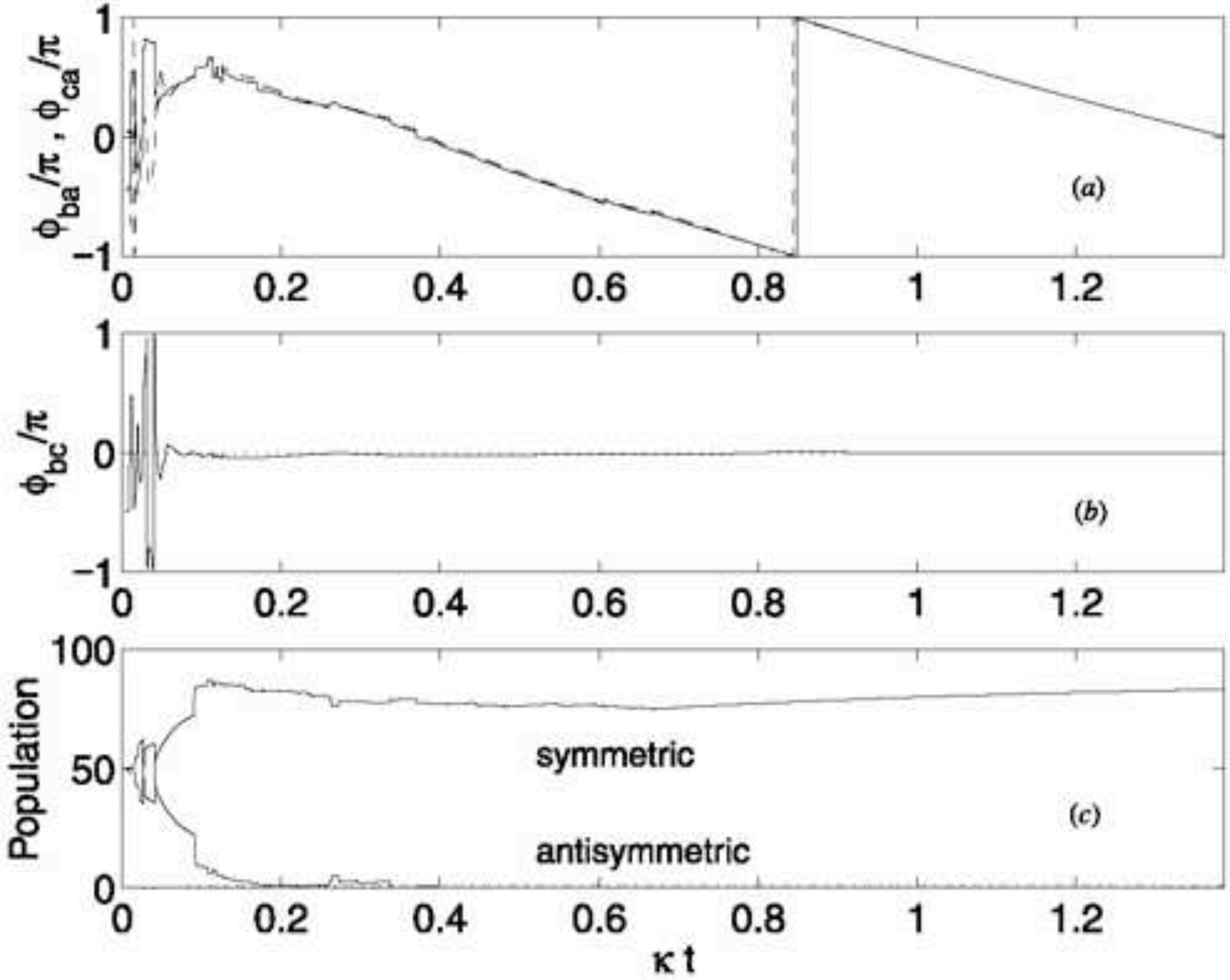
$$|50\rangle_a|50\rangle_b|50\rangle_c$$

Results:

Numerical simulation

$$\frac{\omega_a}{\Gamma} = 5$$

$$\omega_b - \omega_a = 40\kappa$$



detect about 30 atoms

$$\phi_{bc}(t) = \phi_{ba}(t) - \phi_{ca}(t)$$

For this single realisation,

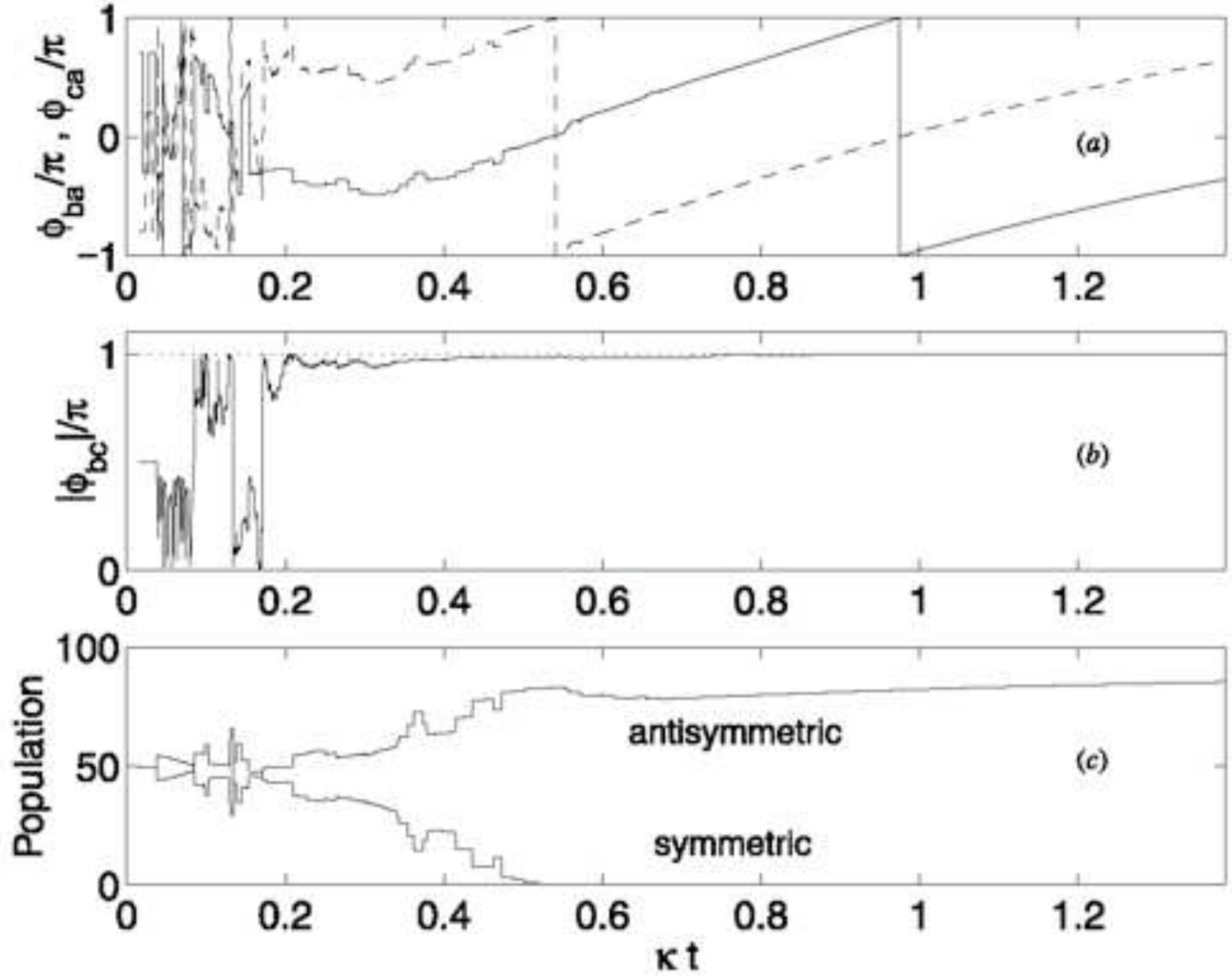
Phase locks to zero phase difference

$$\psi_{\text{sym}} = \psi_b + \psi_c$$

Symmetric superposition of trap modes

Results:

Numerical simulation



$$\phi_{bc}(t) = \phi_{ba}(t) - \phi_{ca}(t)$$

For this single realisation,

Phase locks to phase difference of π

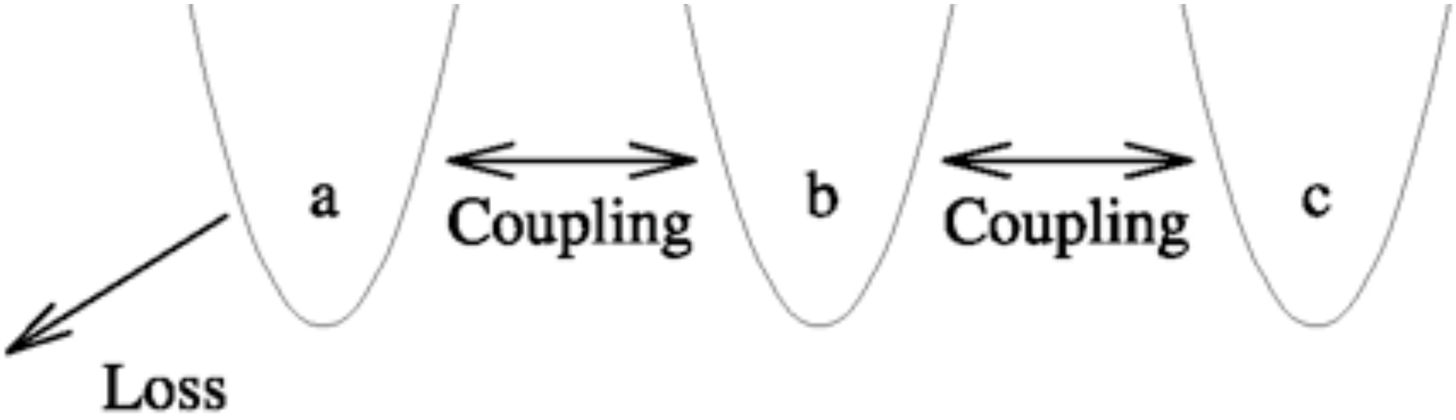
$$\psi_{\mathrm{asym}} = \psi_b - \psi_c$$

Antisymmetric superposition of trap modes

In each case, measurement causes relaxation to eigenstate

Phase Locking (ii)

(with dissipation)

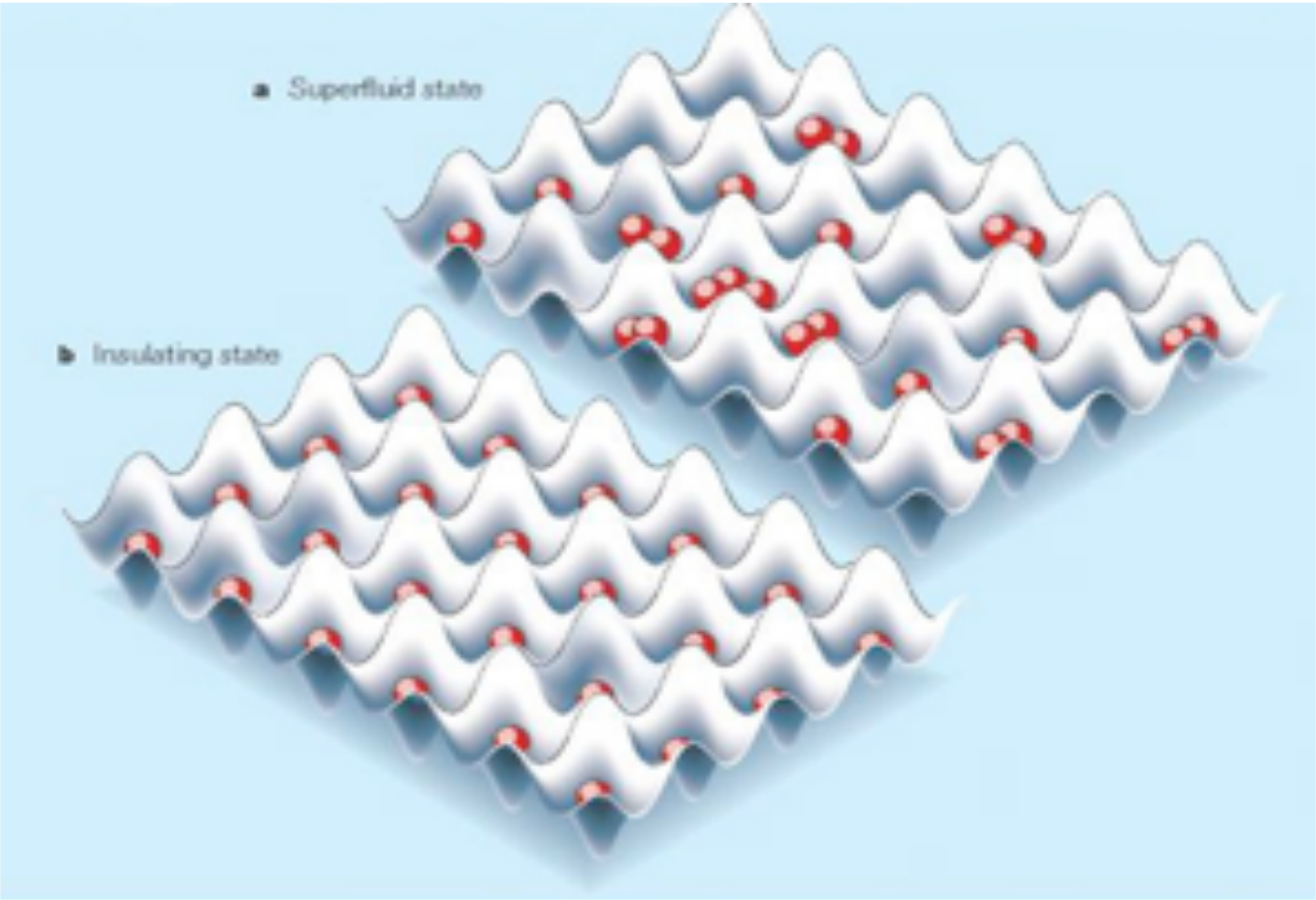


Relevant to output coupler experiment of Kasevich

- Dissipation (rather than measurement) can cause phase locking
- Every trajectory settles to the same phase difference
- Still can't know which mode the lost atom came from (due to coupling)
- Nonlinear interactions also assist in establishing phase
- Same phase locked in regardless of initial state

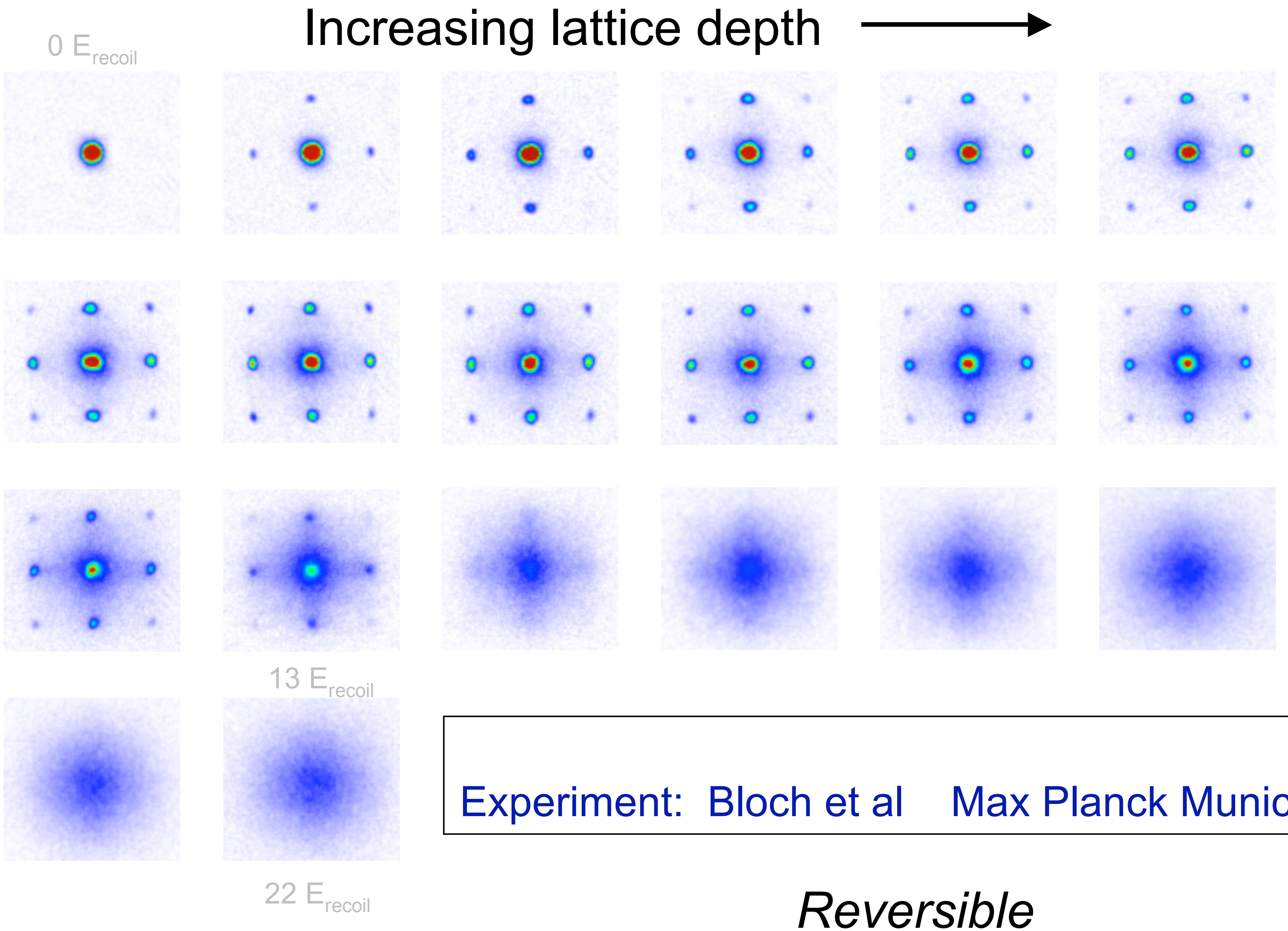
The Mott Insulator Experiment

Greiner, …, Bloch, Nature 415, 29, (2002)



See also Hadzibabic, …, Dalibard, condmat 0405113

Momentum Distribution for Different Potential Depths



Experiment: Bloch et al Max Planck Munich 2001

Reversible