Rotating Bose-Condensates and Vortices

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This Talk

- Rotational and irrotational flow
  - velocity in a superfluid
  - quantised phase circulation
- Some basic vortex properties
  - spatial size, energies
- Vortex formation by mechanical disturbance
  - rotational stirring
  - critical angular velocities
- Vortex lattices
  - role of dissipation
  - dynamics of formation
  - critical angular velocities revisited

Rotating Superfluids

A central topic for Helium superfluids

Defining characteristics:

- Resist being put into rotational motion

- Liquid Helium in a container rotating at sufficient angular velocity, forms vortices, and vortex lattices

- Analogies to superconductors

Rotating Normal fluids

Normal fluids ***allow***

(i) rotational flow - like rigid bodies

$$\mathbf{v}_{\mathrm{sb}} = \boldsymbol{\Omega} \times \mathbf{r}$$

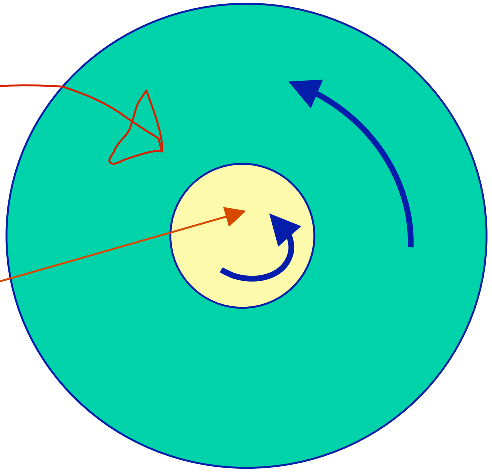
[*vorticity* $\nabla \times \mathbf{v}_{\mathrm{sb}} = 2\boldsymbol{\Omega}$]

(ii) irrotational flow $\nabla \times \mathbf{v} = 0$

$$v \sim 1/r$$

Vortex in normal fluid

irrotational flow



Vortex core - rotational flow

These vortices must ***dissipate***

Classically, rotational flow has lower energy for given angular momentum

Vortices in Superfluids

Condensate order parameter $\psi(\mathbf{r},t) = |\psi(\mathbf{r},t)|e^{iS(\mathbf{r},t)}$ (1)

(e.g. one particle wavefunction)

Superfluid velocity $$\mathbf{v}_s(\mathbf{r},t) \equiv \frac{\hbar}{m}\nabla S(\mathbf{r},t)$$ phase gradient

Can see from current density

$$\mathbf{j}(\mathbf{r},t) = \frac{\hbar}{2mi}\left[\psi^*\nabla\psi - (\nabla\psi^*)\psi\right]$$

Which (using (1) becomes

$$\mathbf{j}(\mathbf{r},t) = n(\mathbf{r},t)\mathbf{v}_s(\mathbf{r},t)$$

with
$$\begin{cases} n(\mathbf{r},t) = |\psi(\mathbf{r},t)|^2 \\ \mathbf{v}_s(\mathbf{r},t) \equiv \dfrac{\hbar}{m}\nabla S(\mathbf{r},t) \end{cases}$$

Notice, since wavefunction is single valued

phase circulation (change of phase over closed path)

$$\oint \nabla S \cdot d\mathbf{l} = 2\pi n \qquad n = 0, \pm 1, \ldots$$ winding number

$$\oint \mathbf{v}_\mathrm{s} \cdot d\mathbf{l} = n\kappa \qquad \textit{Ohnsager–Feynman quantization rule}$$

κ is the **quantum of circulation** $\kappa \equiv \frac{h}{m}$

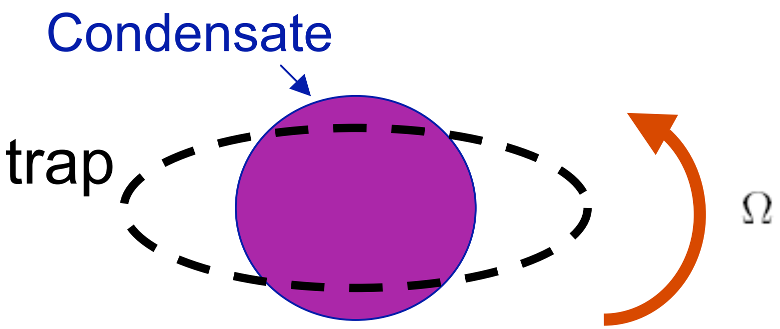
Superfluid velocity automatically irrotational

$$\nabla \times \mathbf{v}_\mathrm{s} = \frac{\hbar}{m} \nabla \times \nabla S = 0$$

Hence cannot support a rigid body motion

But may have angular momentum

Slowly rotating elliptical trap



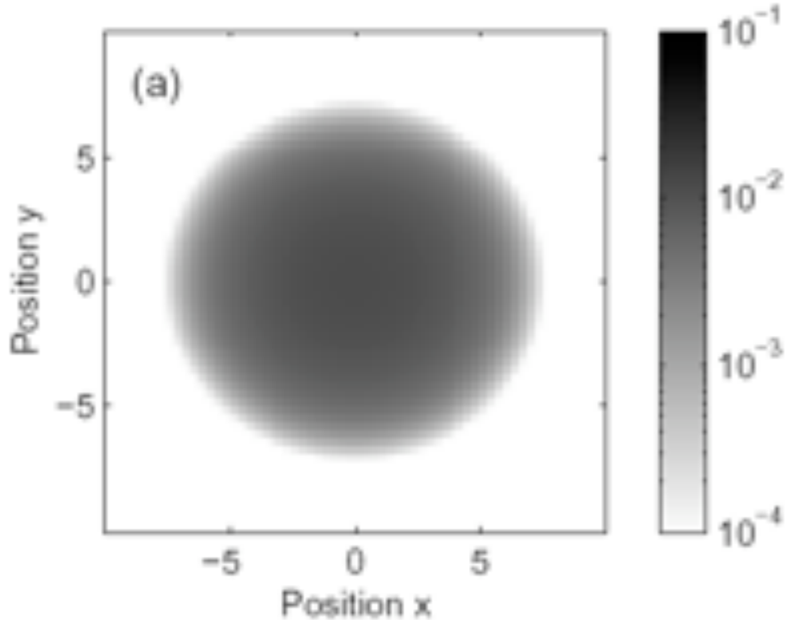
Vortex free case

Condensate is not rotating

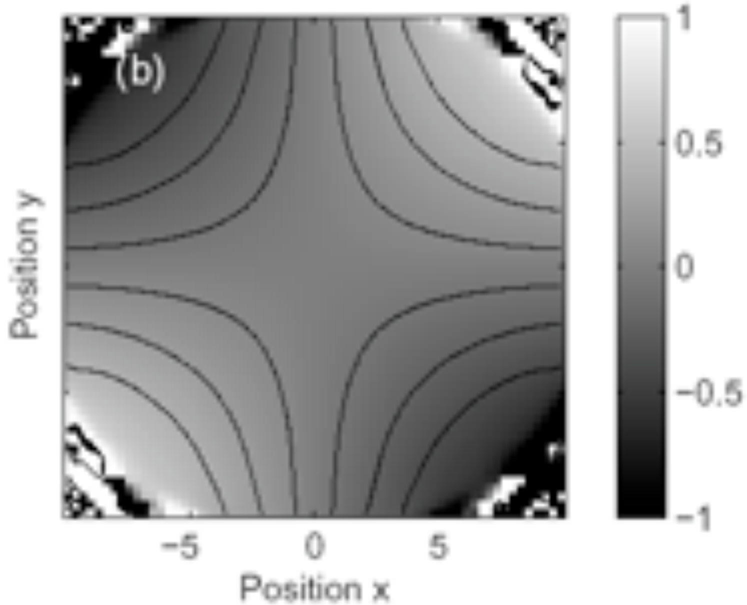
Angular momentum **NOT** zero

$$\langle \hat{L}_z \rangle = 0.0104\,\hbar$$

density



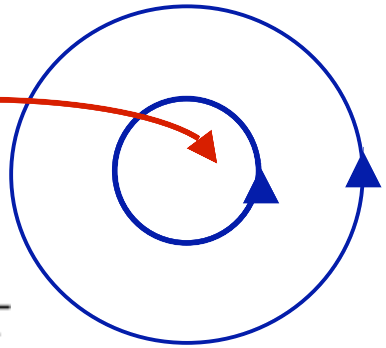
phase



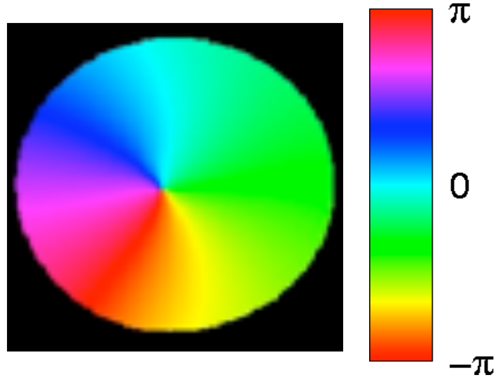
More usual case

Condensate will form a **vortex**

$$|\psi(\mathbf{r},t)| = 0$$



$$\Delta S = 2n\pi$$



Phase

- Vortex is a topological defect
- It's a singularity that carries vorticity $\nabla \times \mathbf{v}_{\mathrm{s}} = n\kappa$

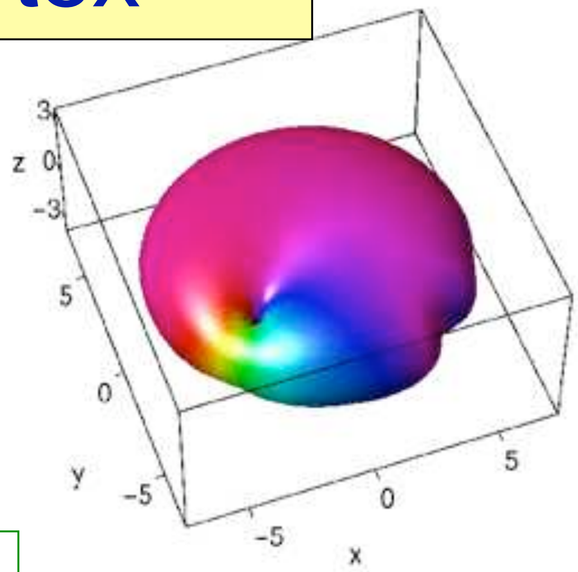
In practice, phase circulation of a vortex is $\pm 2\pi$

(higher winding numbers unstable, we'll see why later)

No dissipation - **persistent flow** - superfluidity

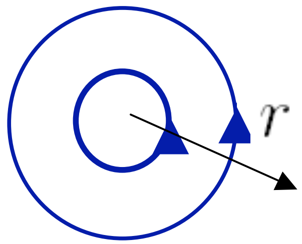
Characteristics of a Vortex

Vortex lives in 3D, and particle density is zero along a line (the vortex line)



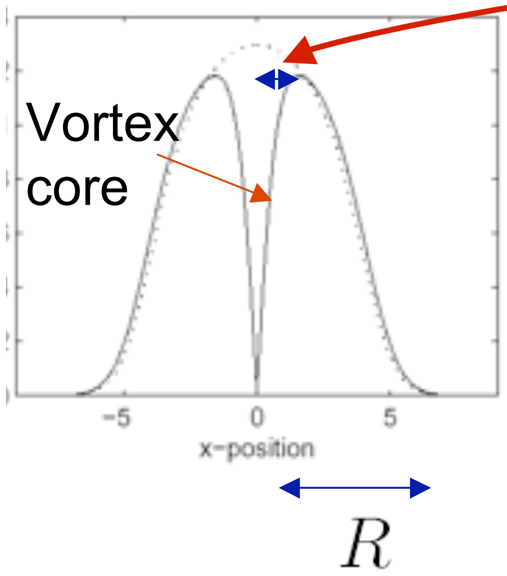
$$\oint \mathbf{v}_s \cdot d\mathbf{l} = 2\pi r v_s(r) = \kappa$$

So $\quad v_s(r) = \dfrac{\kappa}{2\pi r}$



Spatial scales

particle density



Healing length ξ

$$\frac{\hbar^2}{2m\xi^2} = \mu \quad ; \quad \mu = m\omega_\perp^2 R^2/2$$

Chemical potential

Trap frequency

$$\frac{\xi}{R} = \frac{\hbar\omega_\perp}{2\mu}$$

Vortex core is small

Energies associated with a vortex

(relative to ground state with same number of particles)

Main contribution is **kinetic energy** of circulating superfluid

Simple estimate for KE per unit length of a vortex line

$$E_v = m\int_{\xi}^{R_0} \pi n_s v_s^2 r dr = \frac{n_s \kappa^2 m}{4\pi} \ln\left(\frac{R_0}{\xi}\right)$$

- Exclude core (small density)
- Cutoff at edge of container,….

n_s Superfluid density

Multiply-quantized vortex

Energy of vortex with circulation $2\pi n$

$$E = n^2 E_v$$

E_v → Energy of vortex with circulation 2π

Thus it is more favorable to have n singly-charged vortices

i.e. multiple vortex is **unstable**

Energies : more accurate treatment

Use **Gross-Pitaevskii equation**

Time dependent (general form)

$$i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t}=\left[-\frac{\hbar}{2m}\nabla^2+V(\mathbf{r})+\underbrace{NU_0|\psi(\mathbf{r},t)|^2}_{\text{interaction energy}}\right]\psi(\mathbf{r},t)$$

N number of particles

U_0 collisional interaction strength

Assume uniform condensate (no trap) $V(\mathbf{r})=0$

Then use a time independent solution in

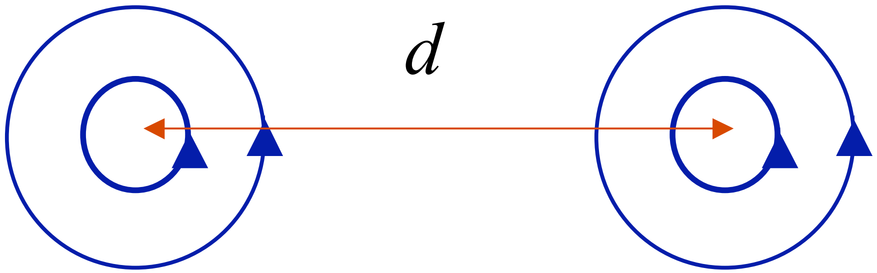
$$E=\int d^3r\left[\frac{\hbar^2}{2m}|\nabla\psi(\mathbf{r})|^2+\frac{1}{2}NU_0|\psi(\mathbf{r})|^4\right]$$

$E_v =$ energy of vortex state – energy of ground state

$$\approx\frac{n_s\kappa^2 m}{4\pi}\ln\left(1.46\frac{R_0}{\xi}\right)$$

(includes interaction energy)

Interaction energy of two vortices

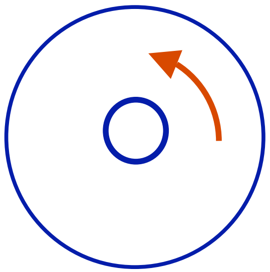


- Calculate expectation energy of two vortex lines separated by distance *d*
- Subtract ground state energy
- Subtract energy $2E_v$ (two individual vortices)

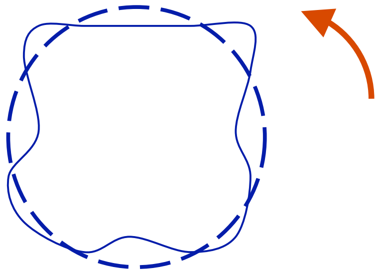
$$E_{int} \approx \frac{n_s \kappa^2 m}{2\pi} \ln\left(\frac{R_0}{d}\right)$$

Angular momentum in condensates

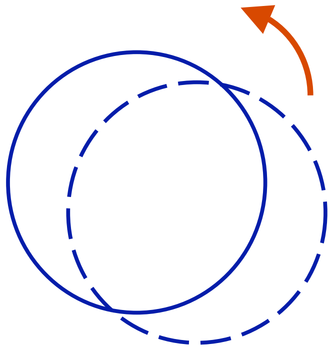
Three ways to carry angular momentum



vortex



Surface mode



Centre of mass motion

How can we create a vortex ?

For topological reasons:

- Vortex must enter from edge of condensate

or

- be created in pairs with opposite 'charge'

Mechanisms for vortex formation

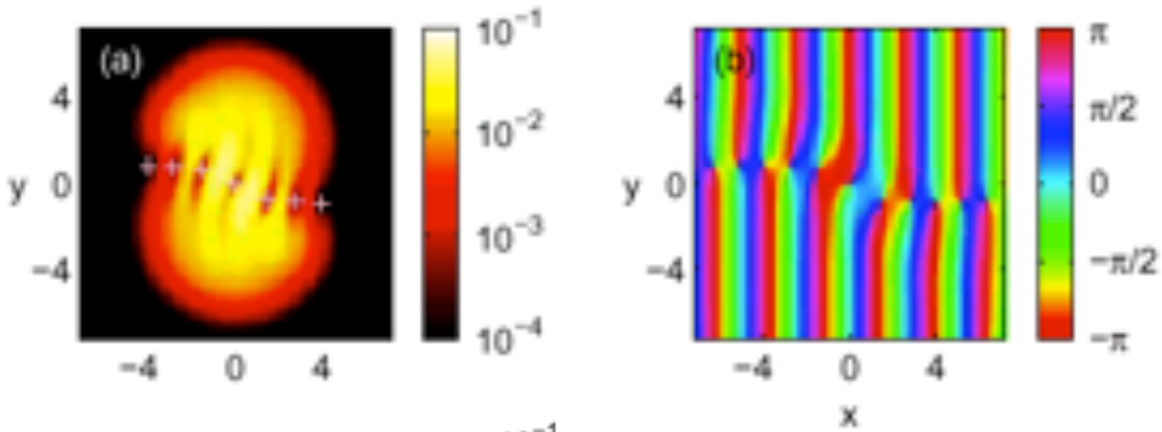
Simulations of the time-dependent GP equation

$$i\,\frac{\partial \psi(\mathbf{r},t)}{\partial t} = -\,\nabla^2 \psi(\mathbf{r},t) + V(\mathbf{r},t)\psi(\mathbf{r},t) + C|\psi(\mathbf{r},t)|^2 \psi(\mathbf{r},t)$$

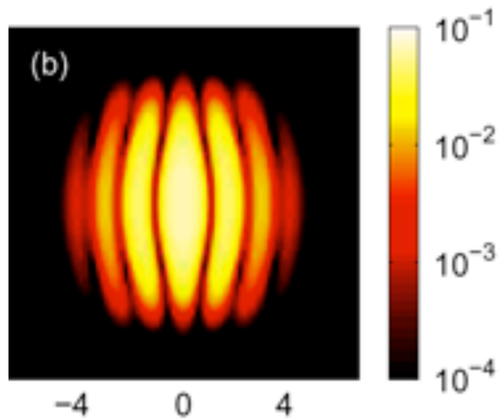
(dimensionless form)

Condensate collisions

Shearing



Direct



Colliding Bose–Einstein Condensates

Rotating collision, $\Delta v = 2$

B. M. Caradoc–Davies, R. J. Ballagh, and K. Burnett*

November 1997

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http://www.physics.otago.ac.nz

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Colliding Bose–Einstein Condensates

Initial separation 10 – the nonlinear regime

B. M. Caradoc–Davies, R. J. Ballagh, and K. Burnett*

September 1997

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Drag an object through condensate

Condensate in axially symmetric harmonic trap

narrow gaussian stirrer (e.g. blue detuned laser)

Stirred Bose–Einstein Condensate

Fast slice with offset 0

B. M. Caradoc–Davies, R. J. Ballagh, and K. Burnett*

November 1997

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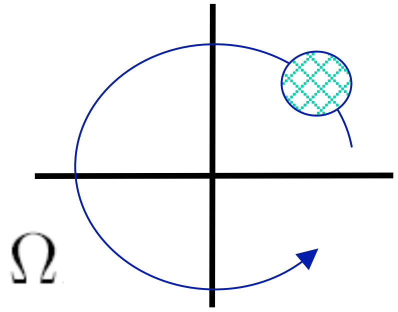
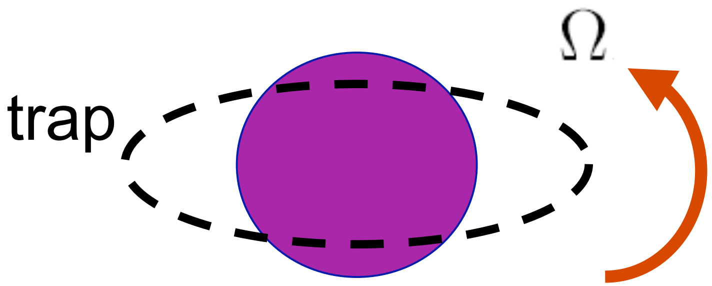
Rotational Stirring

- Condensate in Harmonic trap
- Add additional potential rotating at angular velocity Ω.

- *narrow gaussian stirrer (e.g. blue detuned laser)*



- *deform trap to elliptical, and rotate*



In all cases, there is a **critical angular frequency** associated with vortex behaviour

(topic of considerable debate)

Some fundamental considerations

Rotating Frame (where potential is constant)

- Conceptually easier in this frame
- Statistical mechanics must be done in this frame

$$H \longrightarrow H' = H - \mathbf{\Omega} \cdot \mathbf{L}$$

Energy in rotating frame
(Landau Lifshitz, Mechanics)

Similar result for free energy

$$F' = F - \mathbf{L} \cdot \mathbf{\Omega}$$

Thermodynamic quantity;
minimised at equilbrium

(at $T=0$, $F = E$)

Vortex **energetically favorable**
when energy in the rotating frame lower than ground state

Conventional estimate for **critical frequency of vortex formation**

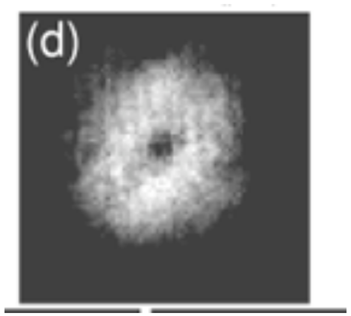
$$E'_{ground} = E'_{vortex} = E_{vortex} - \Omega L_z$$

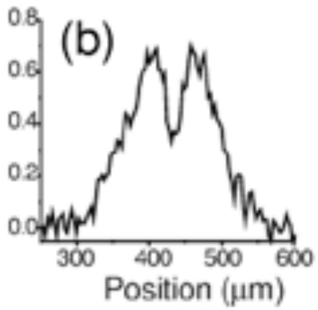
$$\Omega_{crit} = \left(E_{vortex} - E_{ground}\right) / L_z$$

Simplest realisation of a single vortex

- Rotate cloud of cold atoms
- Evaporatively cool
- Condensate forms in vortex state

(Madison, …, Dalibard, PRL **84**, 5 (2000))





Considerable debate about critical frequency measured

- complicated somewhat by trap geometry (rotating ellipse)

- roughly in accord with Landau criteria for superfluid critical velocity

$$\Omega_c = \min\left(\frac{\omega_l}{l}\right)$$

Stringari et al

Rotational stirring of condensate

(Caradoc-Davies, Ballagh, Burnett, PRL **83**,895 (1999))

$$i\,\frac{\partial\psi(\mathbf{r},t)}{\partial t} = -\,\nabla^2\psi(\mathbf{r},t) + V(\mathbf{r},t)\psi(\mathbf{r},t) + C|\psi(\mathbf{r},t)|^2\psi(\mathbf{r},t)$$

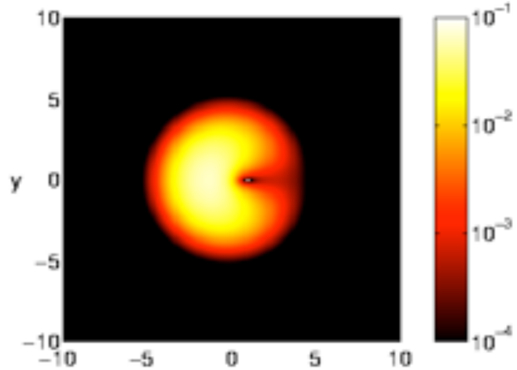
$$V(\mathbf{r},t) = V_{trap} + V_{stir}$$

Simplest conceptual picture:
stirring potential mixes $l=0$ *and* $l=1$ *state*

$$\psi(\mathbf{r},t) = a_s(t)\phi_s(r,n_v) + a_v(t)\phi_v(r,n_v)e^{i\theta},$$

$$n_v = |a_v|^2$$

e.g. $(\phi_v + \phi_s)/\sqrt{2}$

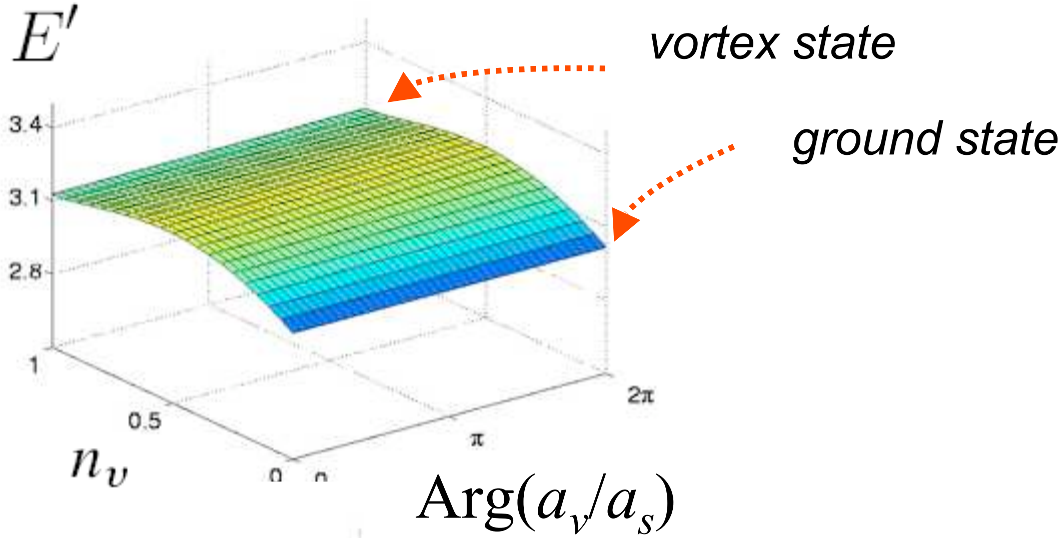


Stirrer geometry important, *require* $\langle\phi_s|V_{stir}|\phi_v\rangle \neq 0$

In rotating frame, energy is conserved

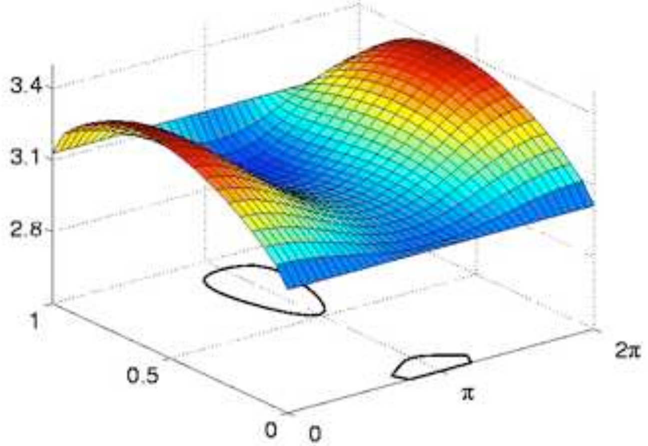
$$E' = E - \Omega L_z$$

System energy as a function of vortex fraction



No stirrer

- energy barrier prevents vortex formation

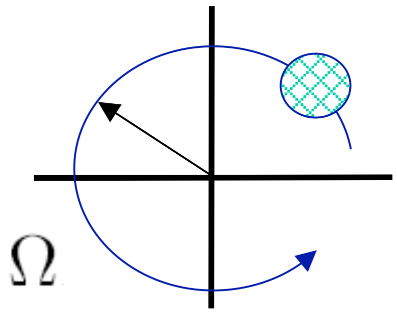


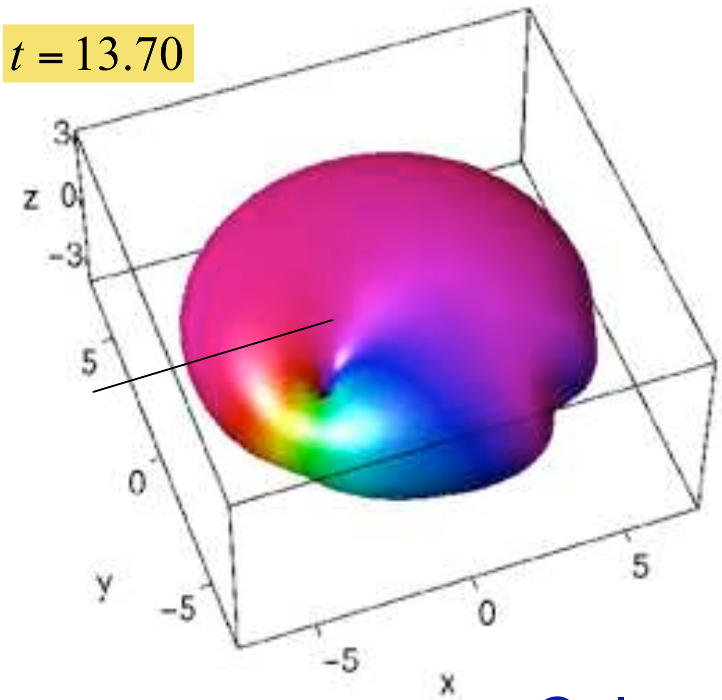
With stirrer

- energy barrier distorted
- can allow cycling to vortex state

Critical stirring : single vortex cycling regime

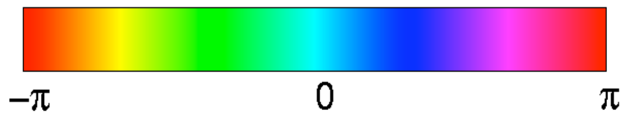
Trap Geometry: TOP





Density isosurface at
$n_s = 10^{-4}$

Colour indicates phase



Rotationally Stirred Bose–Einstein Condensate
Single Vortex Cycling Regime

B. M. Caradoc–Davies, R. J. Ballagh, and P. B. Blakie

Phase on a probability density isosurface ($|\psi|^2 = 10^{-4}$)

$C = 1000, \lambda = 2.828, W_0 = 4, w_s = 4, \rho_s = 2, \omega_t = 0.3$

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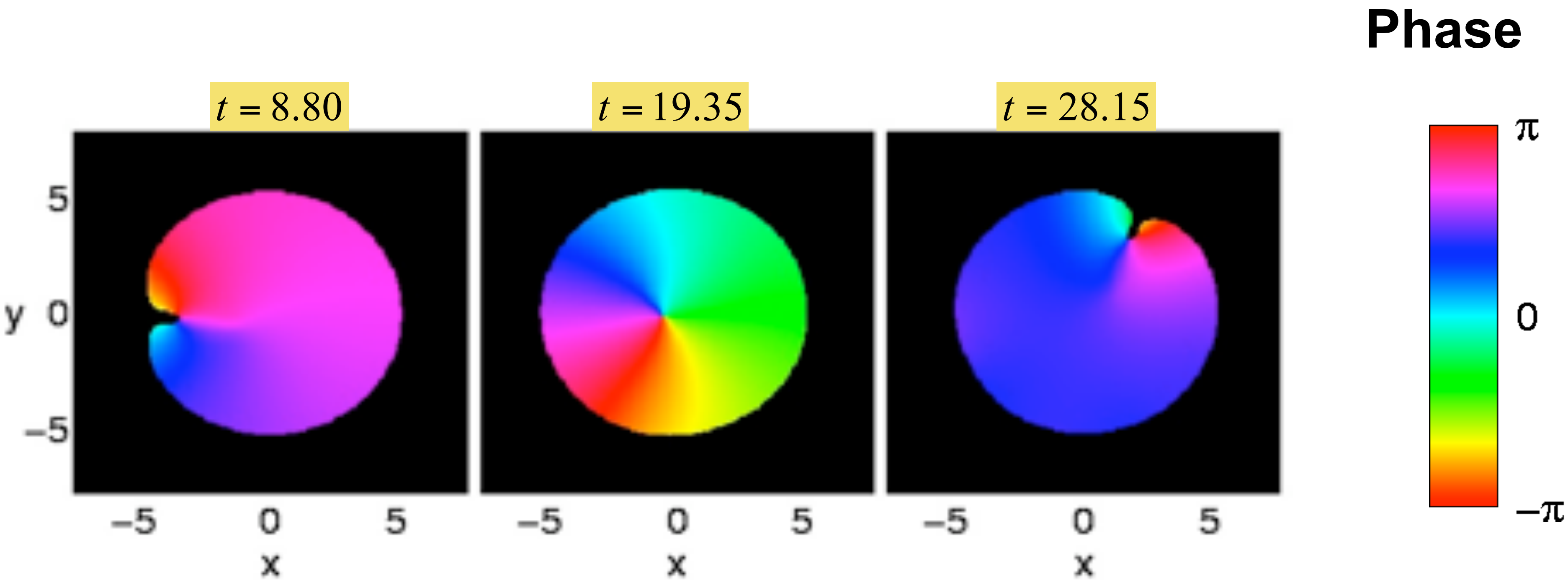
Phase of cycling single vortex

2D

Critical stirring



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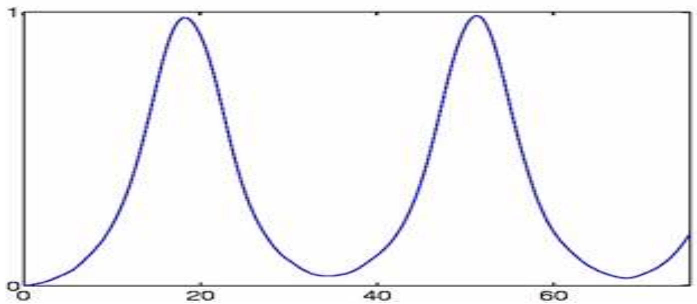
Remarks

1. This is **coherent vortex dynamics** (no dissipation)
   (analog of Rabi cycling for two-state atom driven by laser)

2. Angular momentum cycles regularly

$\langle L \rangle$ $\hbar$



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3. At critical value of stirring frequency, Ω_c
   vortex cycles to centre of condensate

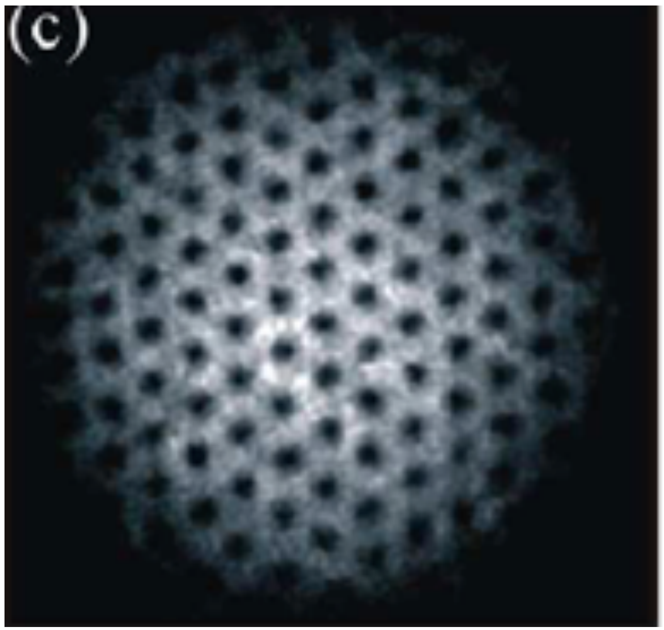
NOTE: Dalibard's experiment can not be explained by this mechanism

his *Vstir* has *l=2* symmetry $\langle \phi_s | V_{stir} | \phi_v \rangle = 0$

Vortex Lattice

(P. Engels,…., E. A. Cornell, Phys. Rev. Lett. **90**, 170405 (2003)

- Rotate cloud of cold atoms, close to trap frequency
- Return potential to cylindrically symmetric
- Evaporatively cool



Vortex lattice **mimics** solid body rotation

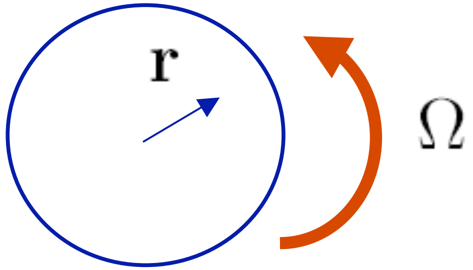
Properties of a Vortex Lattice

- Vortices form a **hexagonal** lattice
  (Abrikosov lattice – known in superconductors and He II)

- Mimics solid body rotation, but irrotational away from vortex cores

For solid body

$$\mathbf{v}_{\mathrm{sb}} = \mathbf{\Omega} \times \mathbf{r}$$



N_{v} vortices

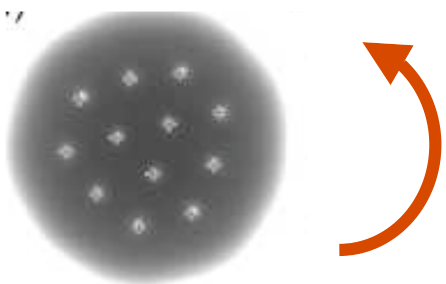
$$\nabla \times \mathbf{v}_{\mathrm{sb}} = 2\mathbf{\Omega}$$

circulation $$\Gamma = \oint \mathbf{v}_{\mathrm{s}} \cdot d\mathbf{l} = 2\Omega A_{\mathrm{v}} \qquad (2)$$

A_{v} enclosed area

For vortex lattice with Nv vortices

$$\Gamma = \oint \mathbf{v}_{\mathrm{s}} \cdot d\mathbf{l} = N_{\mathrm{v}}\kappa \qquad (3)$$



equate (2) and (3) $$n_{\mathrm{v}} = \frac{N_{\mathrm{v}}}{A_{\mathrm{v}}} = \frac{2\Omega}{\kappa}$$

Vortex density depends only on Ω

Dynamics of vortex lattice formation

Vortex lattice is stationary solution of (rotating frame) GPE

$$i\hbar\frac{\partial\Psi_{\mathrm{R}}}{\partial t} = \left\{-\frac{\hbar^2}{2m}\nabla^2 + V_{\mathrm{T}} + U_0|\Psi_{\mathrm{R}}|^2 - \mathbf{\Omega}\cdot\mathbf{L}\right\}\Psi_{\mathrm{R}} \tag{4}$$

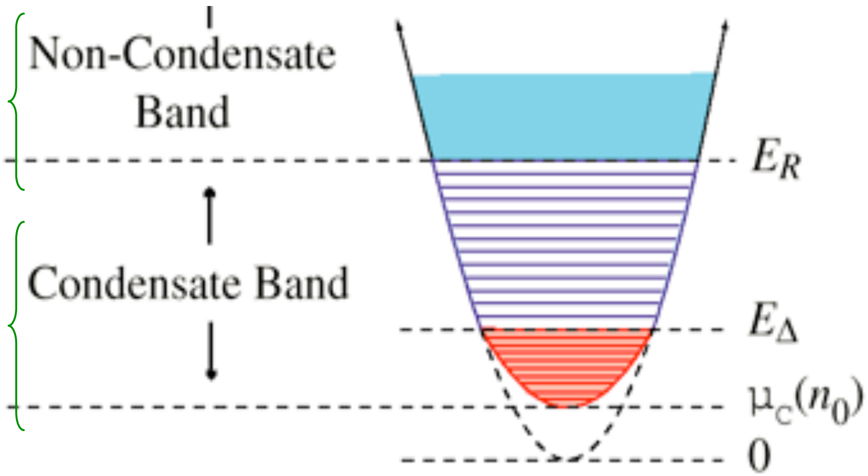
But there is an **energy barrier** to pass through (GPE conserves energy) Hence need **new equation** (including dissipation)

Motivate from Gardiner's *condensate growth equation*

$$\dot{N} = 2\frac{W^+(N)}{k_{\mathrm{B}}T}\left(\mu_{NC} - \mu(N)\right)N \equiv 2(\Gamma_{\mathrm{g}} - \Gamma_{\mathrm{l}})N$$

bath

system



Key quantities:

chemical potentials

bath μ_{NC}

condensate $\mu(N)$

$$W^+ \approx g\frac{4m(ak_{\mathrm{B}}T)^2}{\pi\hbar^3}$$

Add loss/gain terms to GPE, with following mapping

$$\text{growth}: \ \dot{N} = 2\frac{W^+}{k_B T} \mu_{NC} N \quad \leftrightarrow \quad i\hbar\dot{\Psi} = i\gamma \, \mu_{NC} \, \Psi$$

$$\text{loss}: \quad \dot{N} = -2\frac{W^+}{k_B T}\mu N \quad \leftrightarrow \quad i\hbar\dot{\Psi} = -i\gamma\mu \, \Psi$$

$$\gamma \equiv \hbar W^+/k_B T$$

Generalise to local chemical potential $\mu\Psi = i\hbar\frac{\partial\Psi}{\partial t}$

Allow bath to rotate with angular velocity α

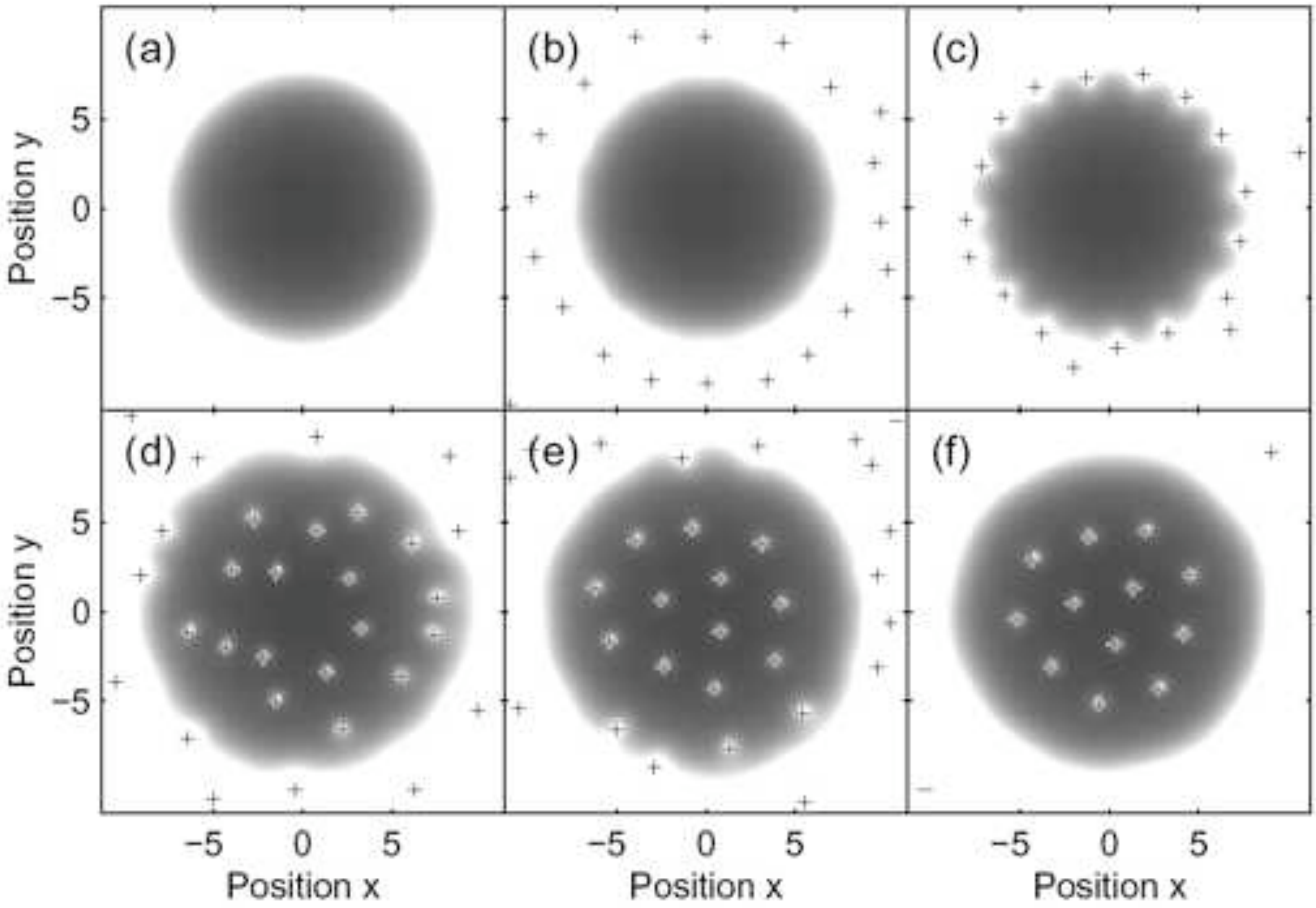
Equation becomes, in lab frame **Thermal GPE**

$$(i-\gamma)\frac{\partial\psi(\mathbf{r},t)}{\partial t} = \left\{-\nabla^2 + V_{\mathrm{T}} + C|\psi(\mathbf{r},t)|^2 + i\gamma\left(\mu_{\mathrm{NC}} + \alpha\hat{L}_{\mathrm{z}}\right)\right\}\psi(\mathbf{r},t)$$

Formulation guarantees equilibrium between bath and condensate $\mu = \mu_{NC}$

with Eq (4) satisfied

Note ; more formal derivation exists, Gardiner, Anglin, Fudge, JPhysB



(Penckwitt, Ballagh, Gardiner, PRL 89, 260402 (2002)

Seeded Vortex Lattice Formation

Thermal cloud rotating with trap

$C = 1000,\ \Omega = \alpha = 0.65,\ \gamma = 0.1,\ u_{NC} = 12$

A. A. Penckwitt, R. J. Ballagh and C. W. Gardiner

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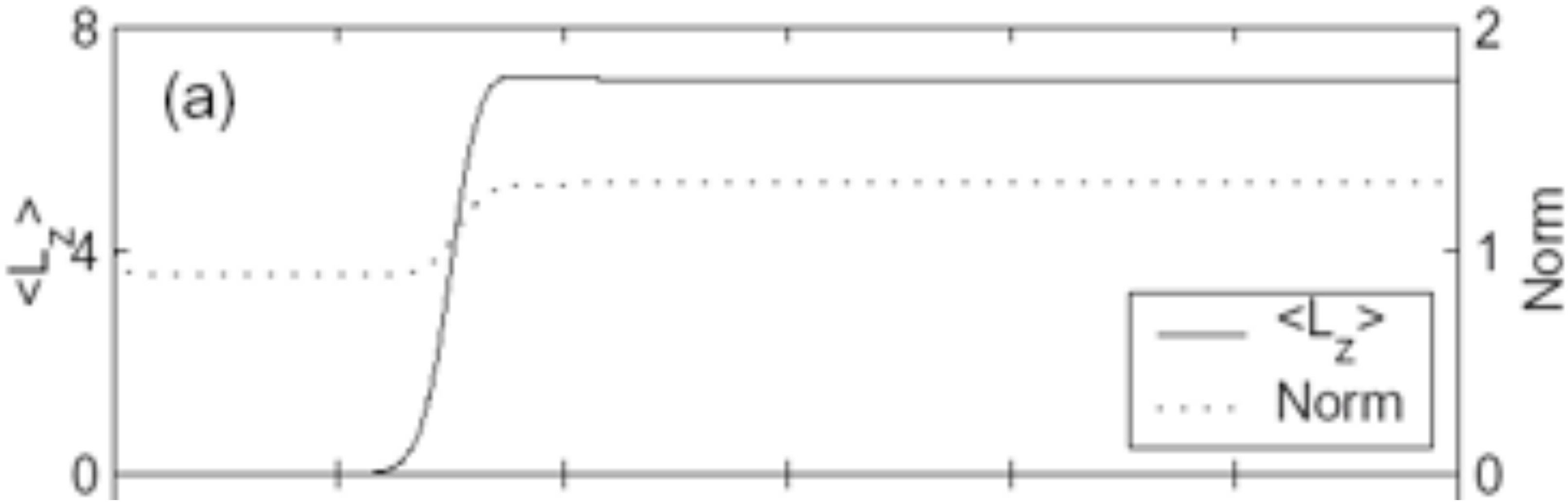
Aotearoa

March 2002



8
(a)
<Lz>
4
0
2
Norm
1
0
<Lz>
Norm

Nucleation of vortices

Make a linearised solution of the Thermal GPE

$$\psi(t) = e^{-i\mu_C t}\left\{\xi_0 + \sum_{n,l} e^{il\phi}[b_{n,l}(t)u_{n,l}e^{-i\epsilon_{n,l}t} + b^*_{n,l}(t)v^*_{n,l}e^{i\epsilon_{n,l}t}]\right\}$$

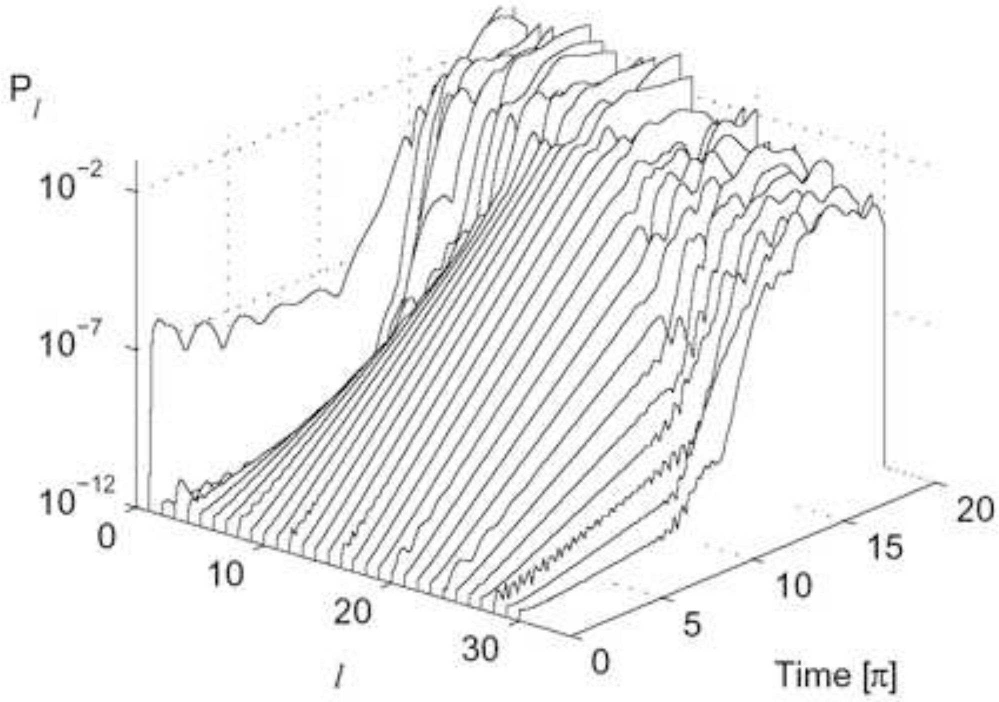
Obtain exponential growth for population of angular momentum components

$$P_{n,l}(t) = P_{n,l}(0)e^{2Gt}$$

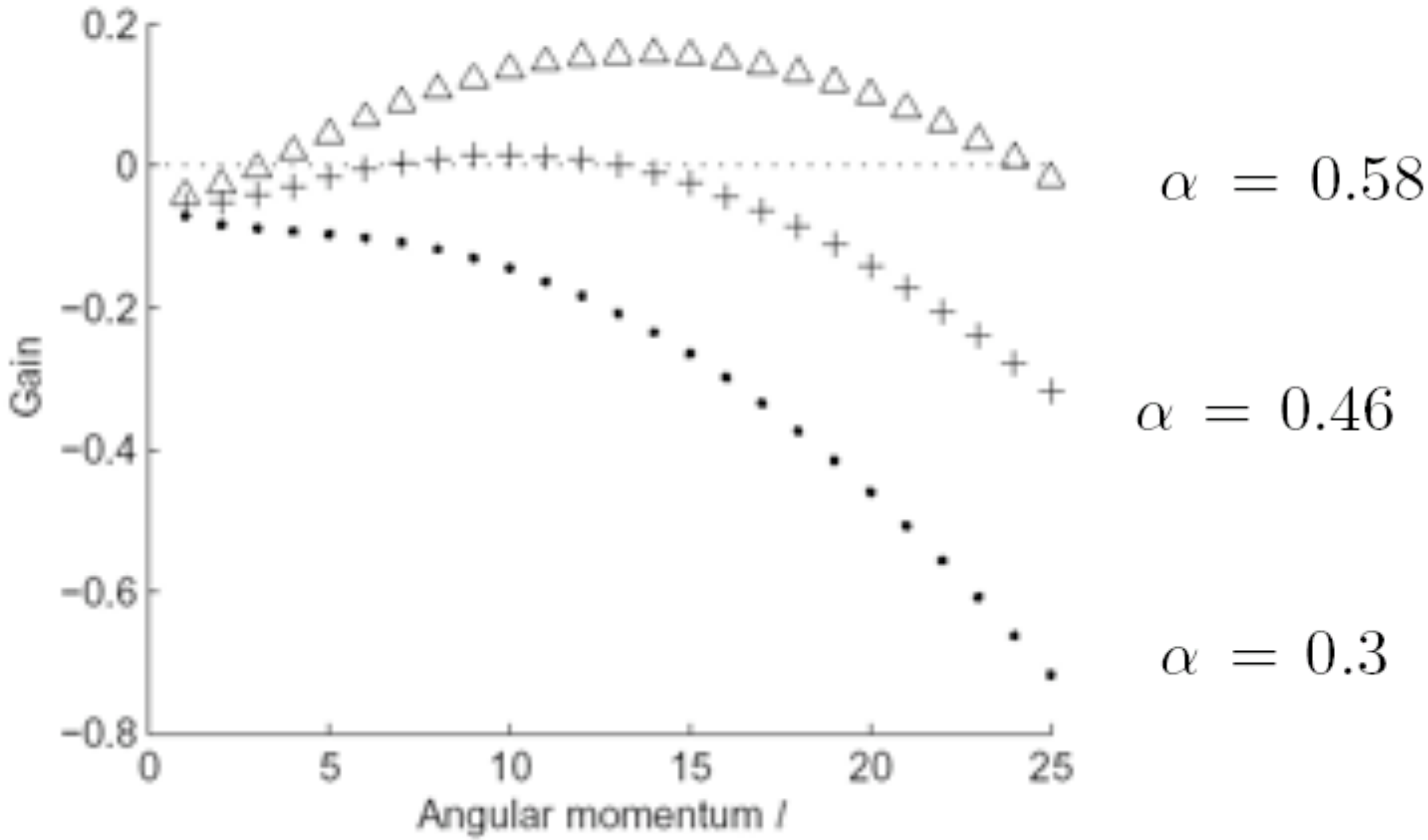
$$G \approx \gamma(\alpha l - \epsilon_{n,l})$$

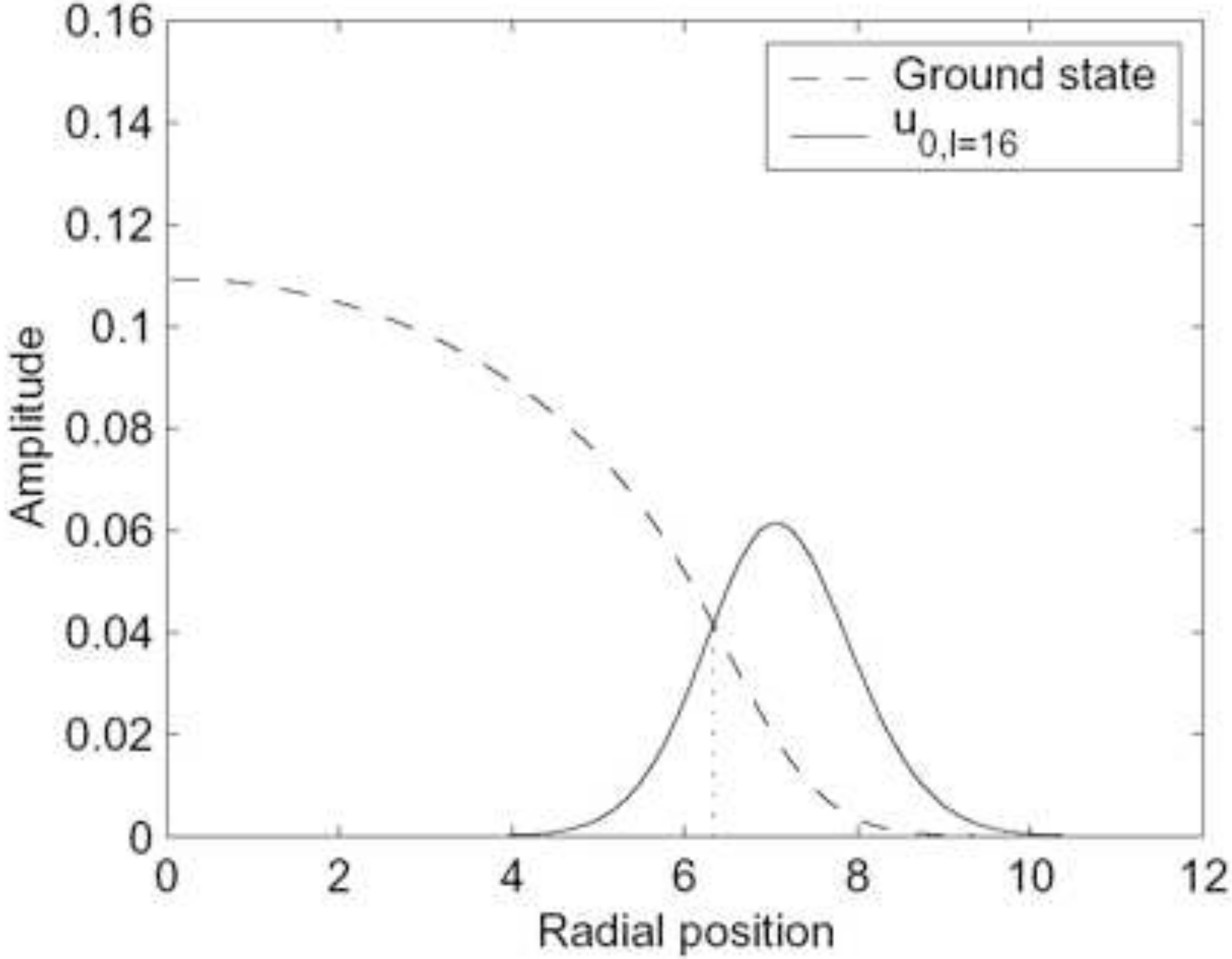
Gives critical velocity

$$\alpha_c = \min\left(\frac{\epsilon_{n,l}}{l}\right)$$



Critical angular velocity




Ground state
$u_{0,l=16}$
Amplitude
Radial position
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0
2
4
6
8
10
12