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Application of the classical field method to acoustic black holes in Bose-Einstein condensates

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Outline

1. What are analogue models of gravity?
   - Hawking Radiation from acoustic black hole
   - BEC's good candidates

2. Configuration for acoustic black hole
   - Laval Nozzle
   - Hydrodynamic approximation

3. Classical field method (extend Hydrodynamic theory)
   - Gross-Piteavskii Equation $\longrightarrow$ ground states, background flow
   - initial quantum noise $\longrightarrow$ vacuum fluctuations

4. Some preliminary results:
   - ground states
   - dynamics

Motivation: Analogue Models

The basic idea: consider fluid flow – Unruh (1981, 1995), Visser (1998)

Continuity: $$\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{v}) = 0$$

Euler's equation: $$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\nabla p}{\rho} + \mathbf{F}$$

Assume fluid is irrotational ($\mathbf{v} = \nabla \phi$), inviscid and barotropic ($p = p(\rho)$) and linearize:

$$\rho \longrightarrow \rho_0 + \rho_1 \qquad \phi \longrightarrow \phi_0 + \phi_1 \qquad p \longrightarrow p_0 + p_1$$

Relativistic wave equation:

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left(\sqrt{-g}\, g^{\mu\nu} \frac{\partial}{\partial x^\nu} \phi_1 \right) = 0$$

with acoustic metric for massless scalar field:

where $$g_{\mu\nu} = \frac{\rho_0}{c} \left(\begin{array}{c|c} -(c^2 - v^2) & -\mathbf{v}^T \\ \hline -\mathbf{v} & \mathbf{I} \end{array} \right) \qquad g = [\mathrm{det}(g^{\mu\nu})]^{-1}$$

Acoustic black holes

- admits acoustic horizons
- Lorentzian geometry with signature (-+++)
- when $\mathbf{v} = 0$ get Minkowski metric for flat space
- No mention of Einstein's field equations

Behaviour of sound waves determined by the acoustic metric.
Line element: $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ (for observer at rest in lab frame)

- supersonic ($v > c$) $\longrightarrow ds^2$ is +ve $\longrightarrow$ spacelike separated – inside horizon
- subsonic ($v < c$) $\longrightarrow ds^2$ is -ve $\longrightarrow$ timelike separated – outside horizon
- transonic ($v = c$) $\longrightarrow ds^2 = 0 \longrightarrow$ sound waves on null geodesics
  $\longrightarrow$ surface defines a horizon

subsonic/supersonic regions are“causally" separated:
$\longrightarrow$ sound waves can be trapped in a flowing fluid!

Careful: we are talking about “apparent" horizons

Acoustic Hawking Radiation

Astrophysical BH: antiparticle + particle pairs formed (vacuum fluctuations) $\longrightarrow$ near event horizon, (-ve E) antiparticle drops into BH whereas (+ve E) particle is radiated.

Analogue of HR in transonic fluid flow first considered by Unruh (1981)
Acoustic BH: quasi-particle pairs (phonons) formed
$\longrightarrow$ energy into non-condensed fraction $\longrightarrow$ reduces kinetic energy of base flow

Basic ingredients for acoustic Hawking Radiation:

- QFT $\equiv$ Vacuum fluctuations
- curved space-time $\equiv$ trapping horizon (event, apparent, ergo-region)

Should observe a thermal spectrum of phonons with Hawking temperature:

$$k_B T_H = \frac{\hbar g_H}{2\pi c}$$

Surface gravity: see Visser (1998)

$$g_H = \frac{1}{2}\frac{\partial(c^2 - v_\perp^2)}{dr} = c\frac{\partial(c - v_\perp)}{dr}$$

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Connection with Bose-Einstein condensates

Fluid dynamics $\longrightarrow$ connection to the GPE:

$$i\hbar\frac{\partial\psi}{\partial t} = \left[-\frac{\hbar^2\nabla^2}{2m} + V(\mathbf{r})\right]\psi + U_0|\psi|^2\psi \qquad U_0 = \frac{4\pi\hbar^2 a}{m}$$

$$\psi(\mathbf{r},t) = \sqrt{n}\exp(i\theta) \qquad n = |\psi|^2 \qquad \mathbf{v} = \frac{\hbar}{m}\nabla\theta$$

Equations of motion for n and θ with **hydrodynamic approximation**
$\longrightarrow$ continuity and Euler's equations where pressure and external force are:

$$p = \tfrac{1}{2}U_0 n^2 \qquad \mathbf{F} = -\tfrac{1}{m}\nabla V_{\text{ext}}(\mathbf{r})$$

Why BEC's?

- They are cold! - Hawking radiation might be observed since $T_H \sim T_c$
  Estimate: $T_H \approx 70$ nK $\sim T_C \approx 90$ nK (Visser (2001))
- Exhibit superfluid flow (inviscid, irrotational)
- Microscopic theory well understood
- Many experimental configurations are possible

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How do we make an acoustic BH?

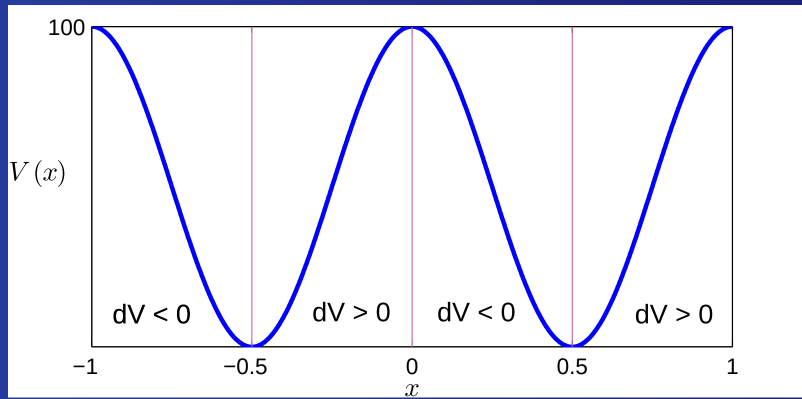
Nozzle – Garay *etal.* (2001)

Laval Nozzle – Barceló *etal.* (2001), Sakagami and Ohashi (2002)

$\longrightarrow$ **AHC from double Laval nozzle formed with external potential**

consider Continuity and Euler's equation with external potential $V(x)$
$\longrightarrow$ gives "nozzle" equation:

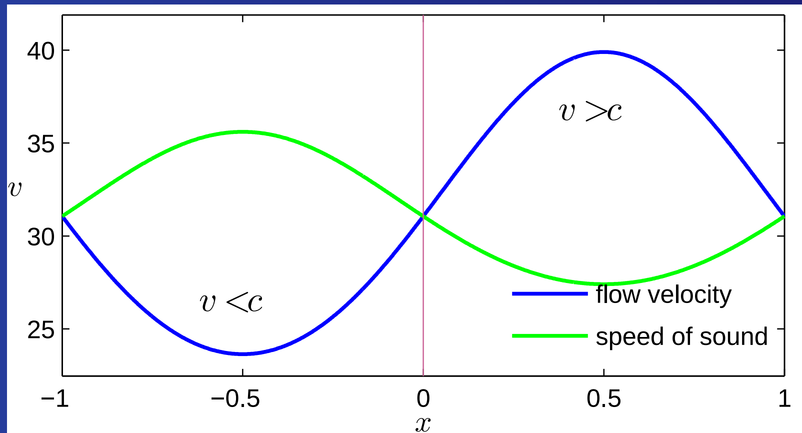
$$\frac{dv}{v} = \left(\frac{c^2}{c^2 - v^2}\right)\frac{dV(x)}{mc^2}$$



Use potential:

$$V(x) = V_0 \cos^2\left(\frac{n\pi x}{2L}\right) \qquad -L \le x \le L$$

with $n = 2$ – need double nozzle for stable flow: subsonic $\longrightarrow$ supersonic $\longrightarrow$ subsonic



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Classical field method

The truncated Wigner Method $\longrightarrow$ GPE + initial noise

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi + U_0 N|\psi|^2\psi$$

Vacuum fluctuations accounted for by initial noise in Wigner represention:

$$\psi(x, t=0) = \psi_{\text{GS}}(x) + \chi(x)$$

amplitudes of virtual particles given by (half particle per mode):

$$\langle\chi_i^*\chi_j\rangle = \frac{1}{2}\delta_{ij}, \qquad \langle\chi_i\chi_j\rangle = 0, \qquad \langle\chi_i^*\chi_j^*\rangle = 0$$

Truncated Wigner method – third order terms in equation of motion for Wigner function neglected. Validity:

- physics given by interaction of highly occupied modes with vacuum modes
- initial noise leads to slight heating – Steel *etal.* (1998) $\longrightarrow$ care required!

Similar formalism has recently been used to predict *quantum turbulence* in colliding condensates – Norrie *etal.* (2004)

Ground states (1D)

Assume stationary state:

$$\psi(x,t) = s(x)\, e^{i\vartheta(x)} e^{-i\mu t/\hbar}$$

with constant current:

$$j = nv = s^2 v$$

time-independent GPE gives:

$$\mu s = -\frac{\hbar^2}{2m}\frac{d^2 s}{dx^2} + V(x)s + U_0 s^3 + \frac{mj^2}{2s^3}$$

Periodic boundary conditions:

$$s(x=-L) = s(x=L) \qquad \Delta\vartheta = \frac{m}{\hbar}\oint v(x)\,dx = 2\pi w$$

Construct a ground state wavefunction:

$$\vartheta(x) = \frac{m}{\hbar}\int \frac{j\,dx}{s(x)^2}$$

$$\psi_{\text{GS}}(x) = s(x)\exp\big(i\vartheta(x)\big)$$

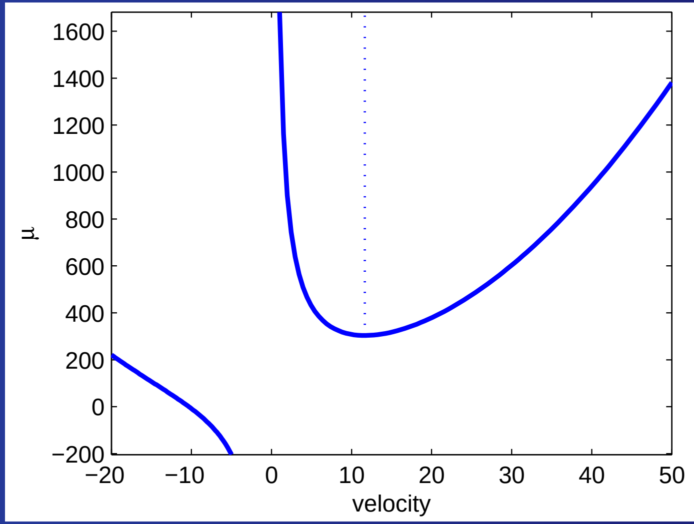
Hydrodynamic approx

Hydrodynamic approximation → drop quantum pressure term → classical fluid

$$\frac{1}{2}v^2 + \frac{C\sqrt{2J}}{v} + V(x) - \mu = 0$$

two solutions are degenerate allowing transonic crossover ($v = c$) when:

$$\frac{8}{27}(V(x) - \mu)^3 + 2JC^2 = 0$$



Crossover at throat of nozzle: $V(x) = V_0$
→ chemical potential: $\mu_{crit} = \mu(J, C, V_0)$

Form transonic solution:

- subsonic branch for $-L \leq x < 0$
- supersonic branch for $0 \leq x < L$

→ use these solutions to find ground states by solving GP Equation. – fix V_0, C, and w

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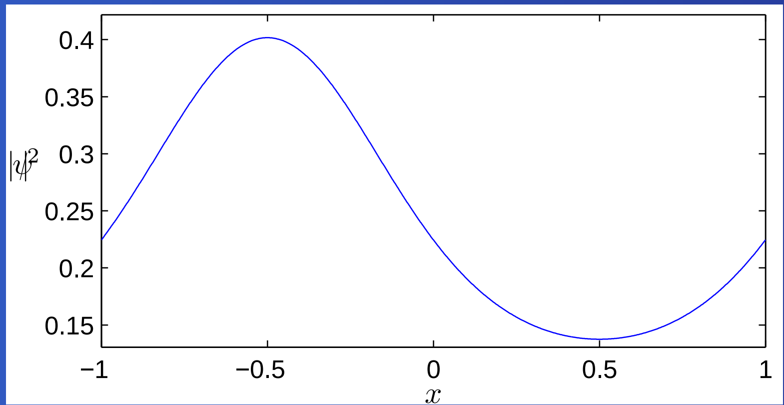
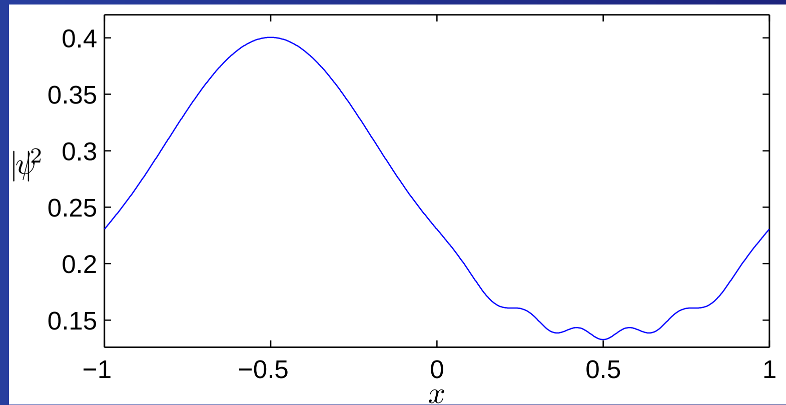
Ground state results: winding number

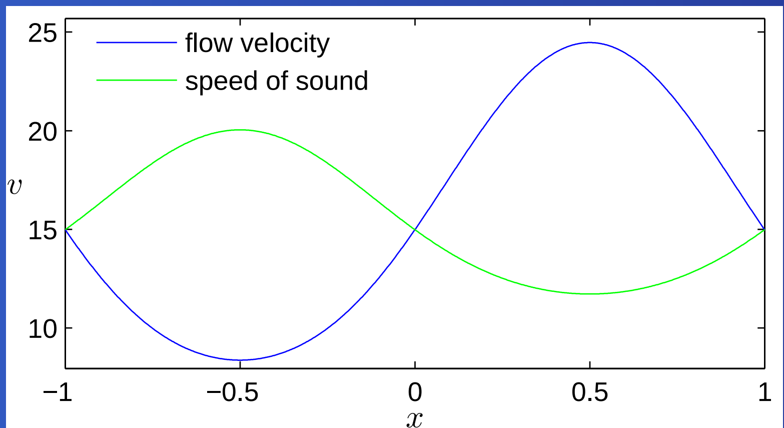
low winding number: $w = 5$, $C = 1000$, $V_0 = 100$



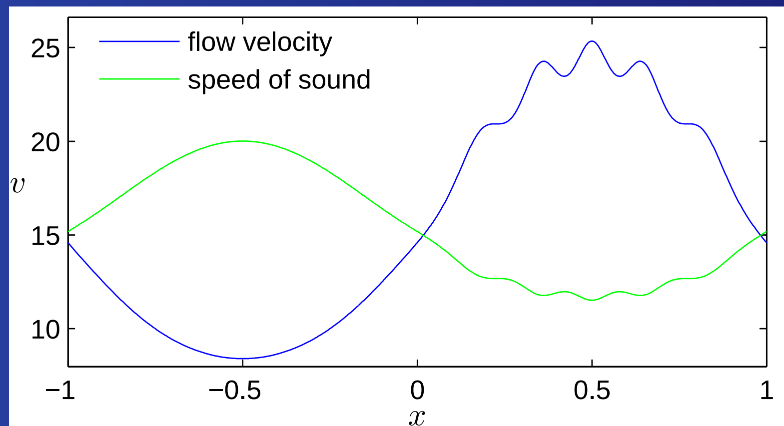
(a) density HD



(b) density GPE



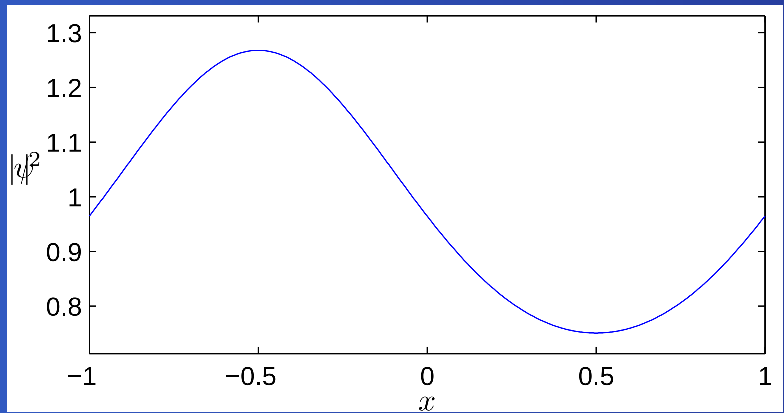
(c) velocity HD



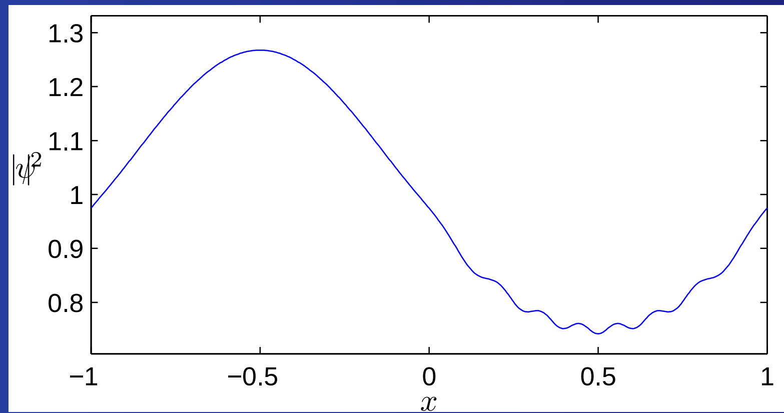
(d) velocity GPE

Ground state results: winding number

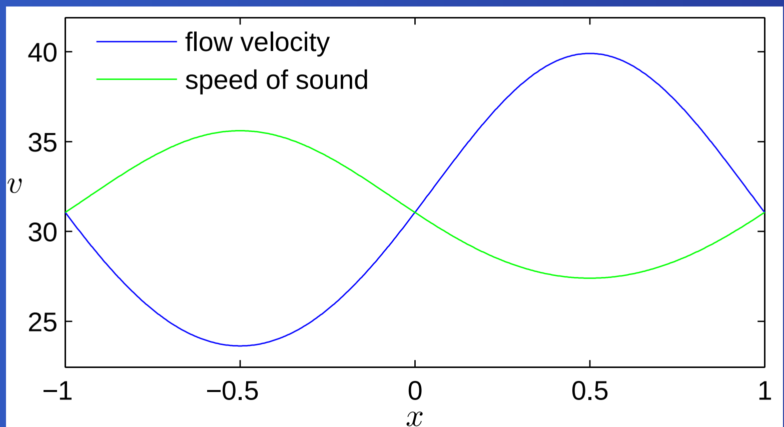
intermediate winding number: $w = 10$, $C = 1000$, $V_0 = 100$



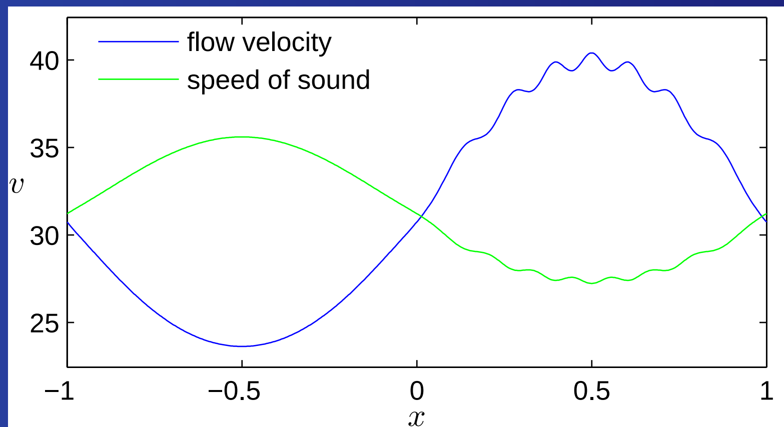
(e) density HD



(f) density GPE



(g) velocity HD



(h) velocity GPE

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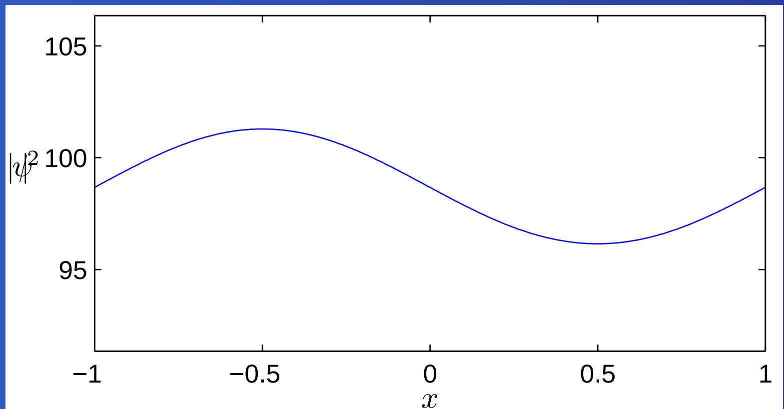
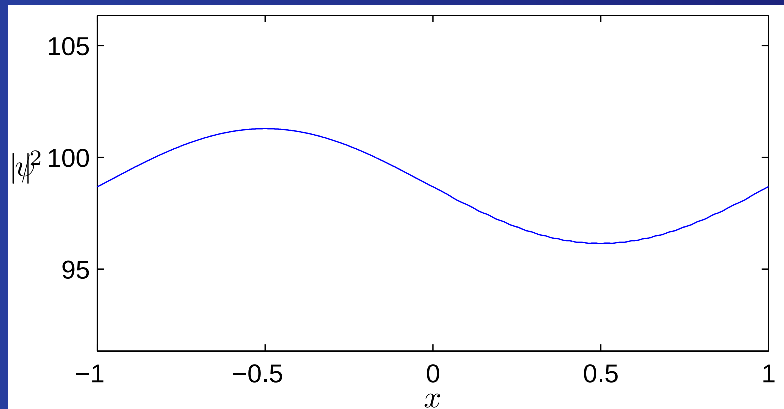
Ground state results: winding number

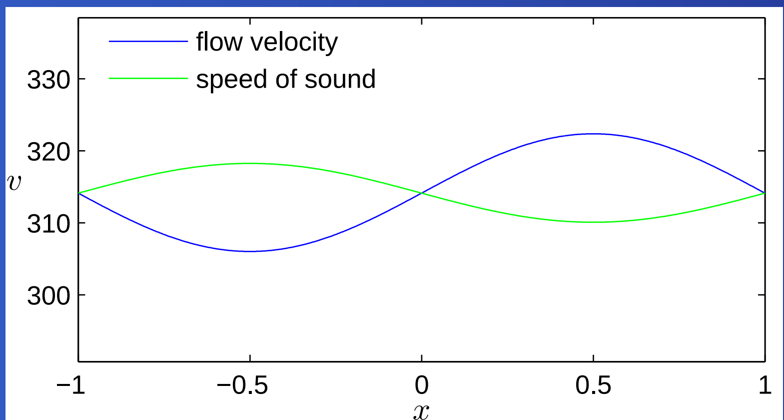
high winding number: $w = 100$, $C = 1000$, $V_0 = 100$



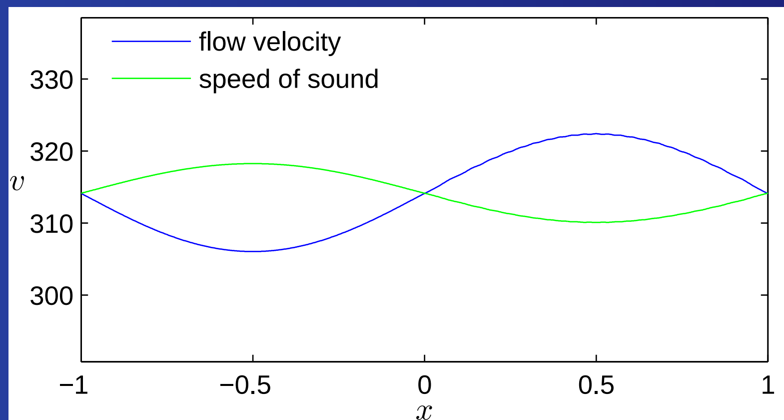
(i) density HD



(j) density GPE



(k) velocity HD



(l) velocity GPE

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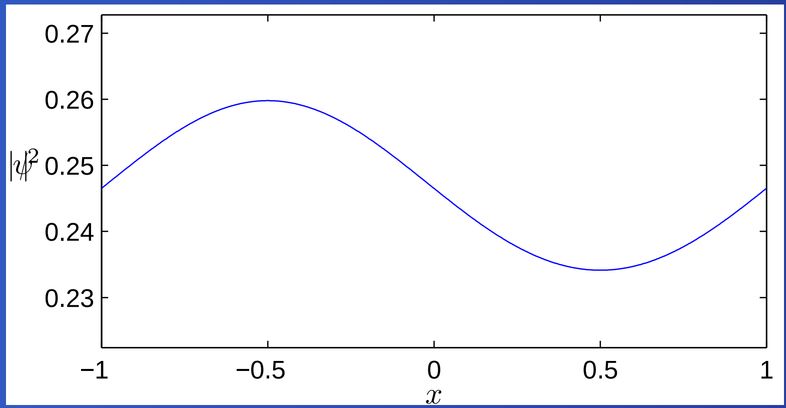
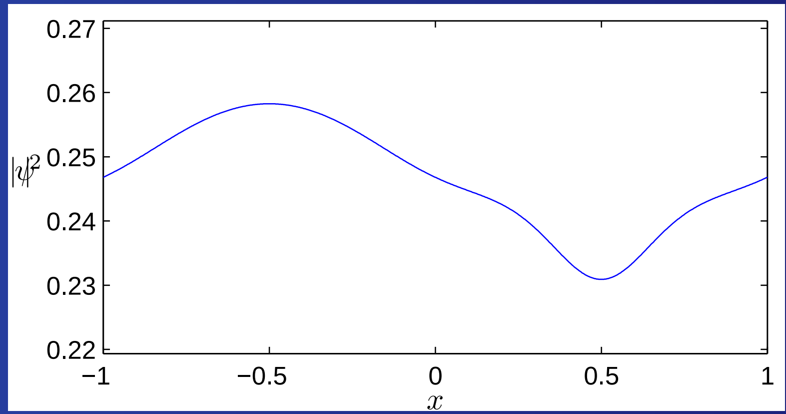
Ground state solutions: V_0 potential

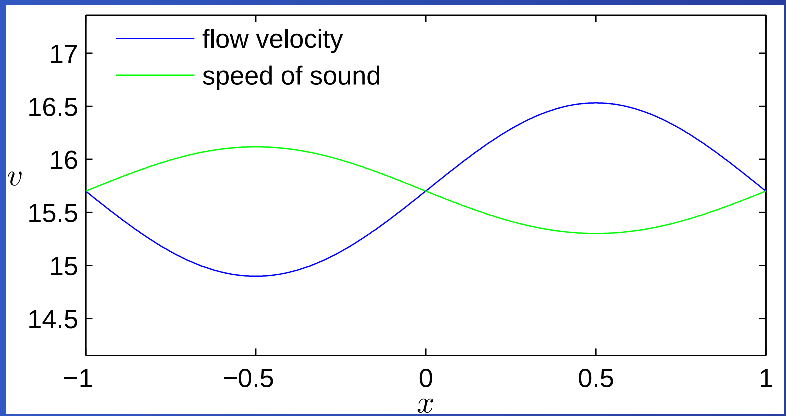
small potential: $w = 5$, $C = 1000$, $V_0 = 1$



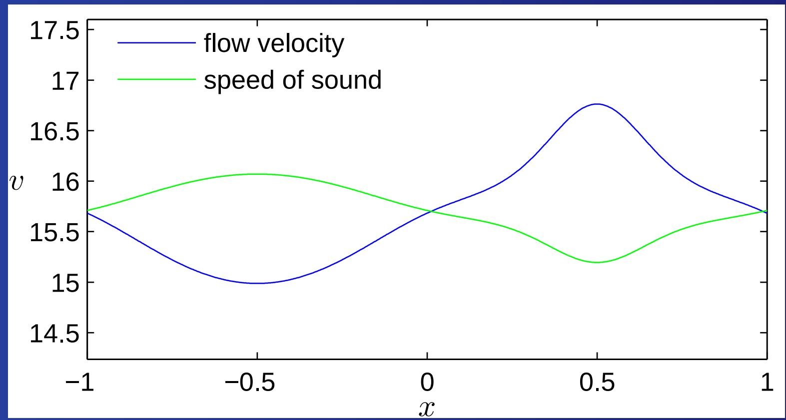
(m) density HD



(n) density GPE



(o) velocity HD



(p) velocity GPE

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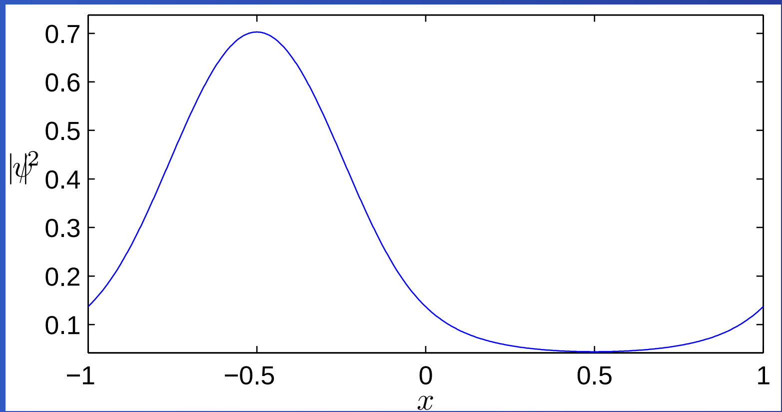
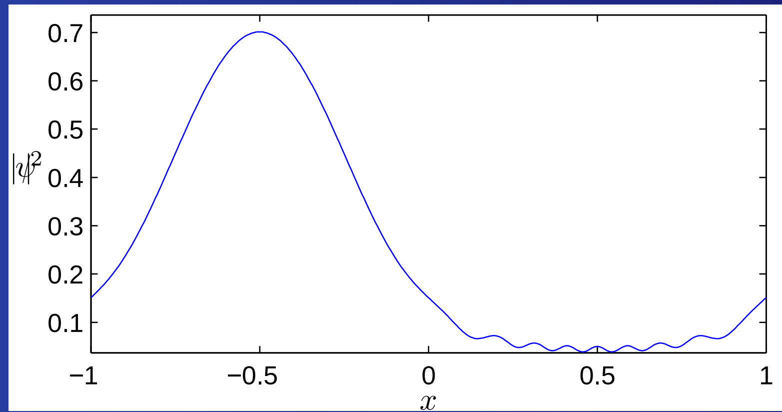
Ground state solutions: V_0 potential

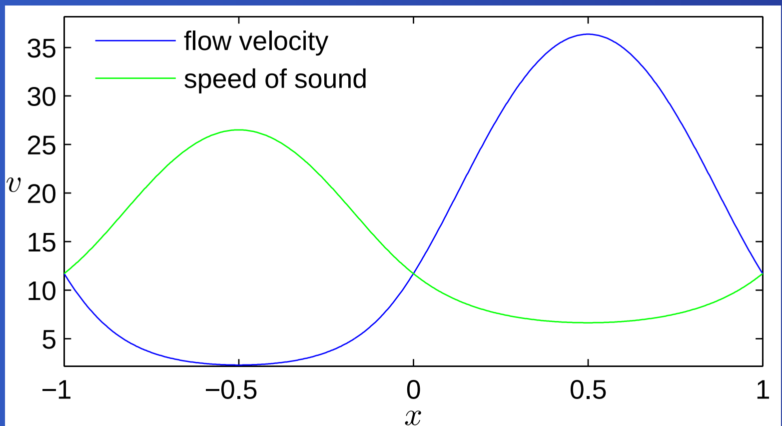
large potential: $w = 5$, $C = 1000$, $V_0 = 500$



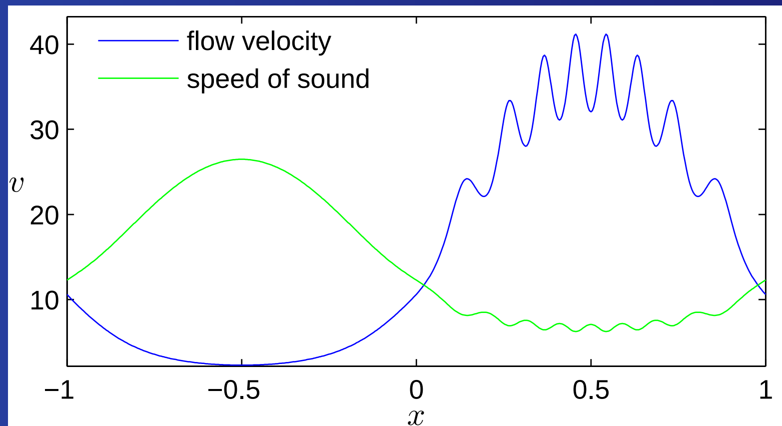
(q) density HD



(r) density GPE



(s) velocity HD



(t) velocity GPE

Dynamics: quantum noise

Movie:

$C_{NL} = 494$, $V_0 = 100$, $w = 5$
$N0 = 1$ initial noise on 200 modes

Implementation details:

- RK4IP algorithm: 4th order Runge-Kutta in interaction picture (Otago group)
- Fast but unstable for modes with large $k \longrightarrow$ may need to use a pseudo-spectral method with a projector in momentum space

Remarks on Hawking radiation

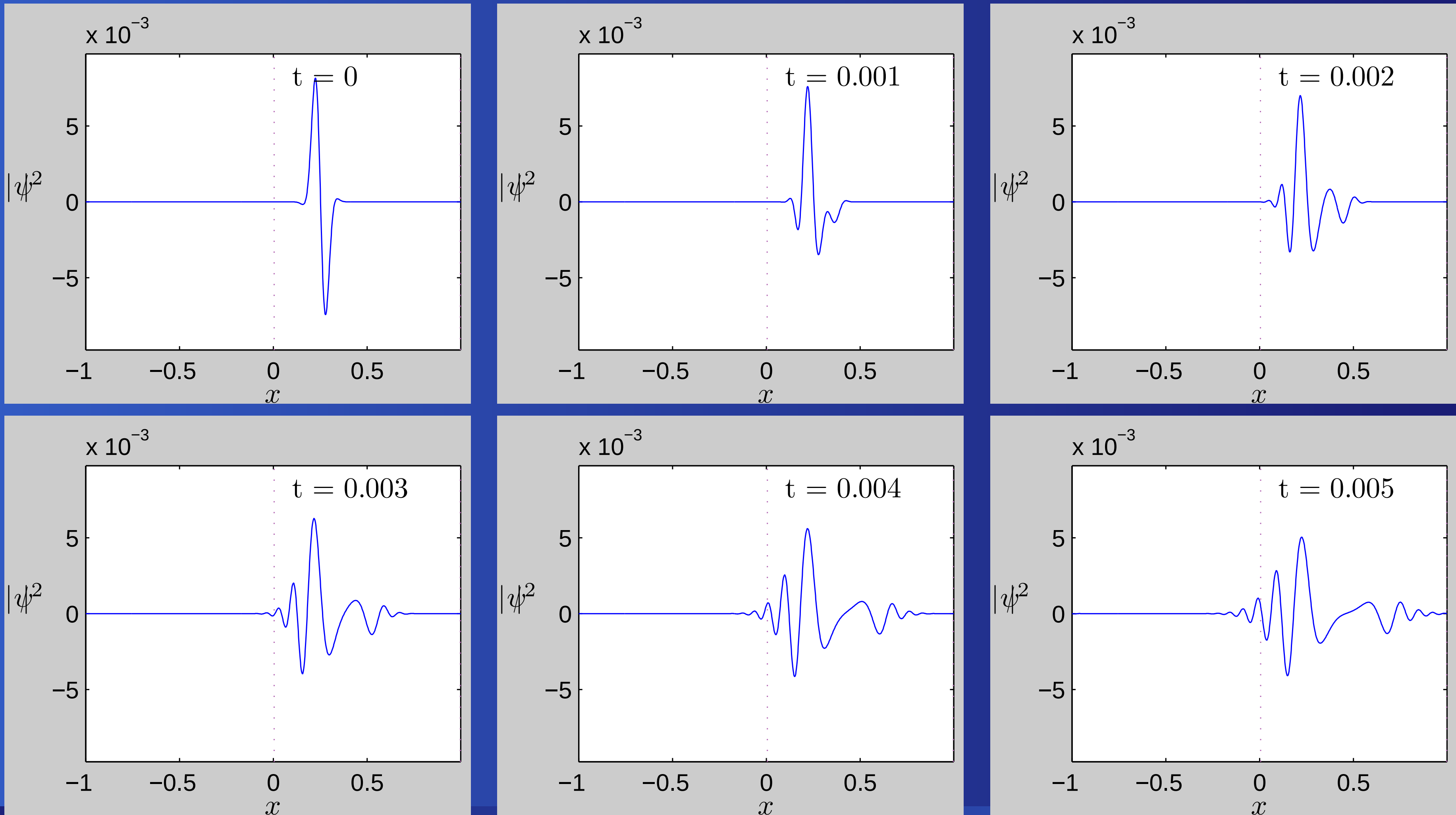
1. Do we have an acoustic horizon?
   - phonon modes (large λ) are attenuated at a BH horizon
   - Phononic modes are Doppler shifted to shorter λ (large k) approaching BH horizon

2. signatures of AHR – what to look for:
   - thermal phonon spectrum with temperature $k_B T_H = \hbar g_H / 2\pi c$

3. Caution:
   - Recall Bogoliubov dispersion relation – only low k modes are phononic; high k modes act as free particles (not governed by acoustic metric)
   - Doppler shifted modes no longer phononic
   - Effect of shifting horizon, ie. fluctuating metric
   - Hawking temperature might be too low to extract from noise

Gaussian wavepacket inside BH horizon

$x_{\text{center}} = +0.25,\ w = 10,\ V_0 = 200,\ C = 1974$



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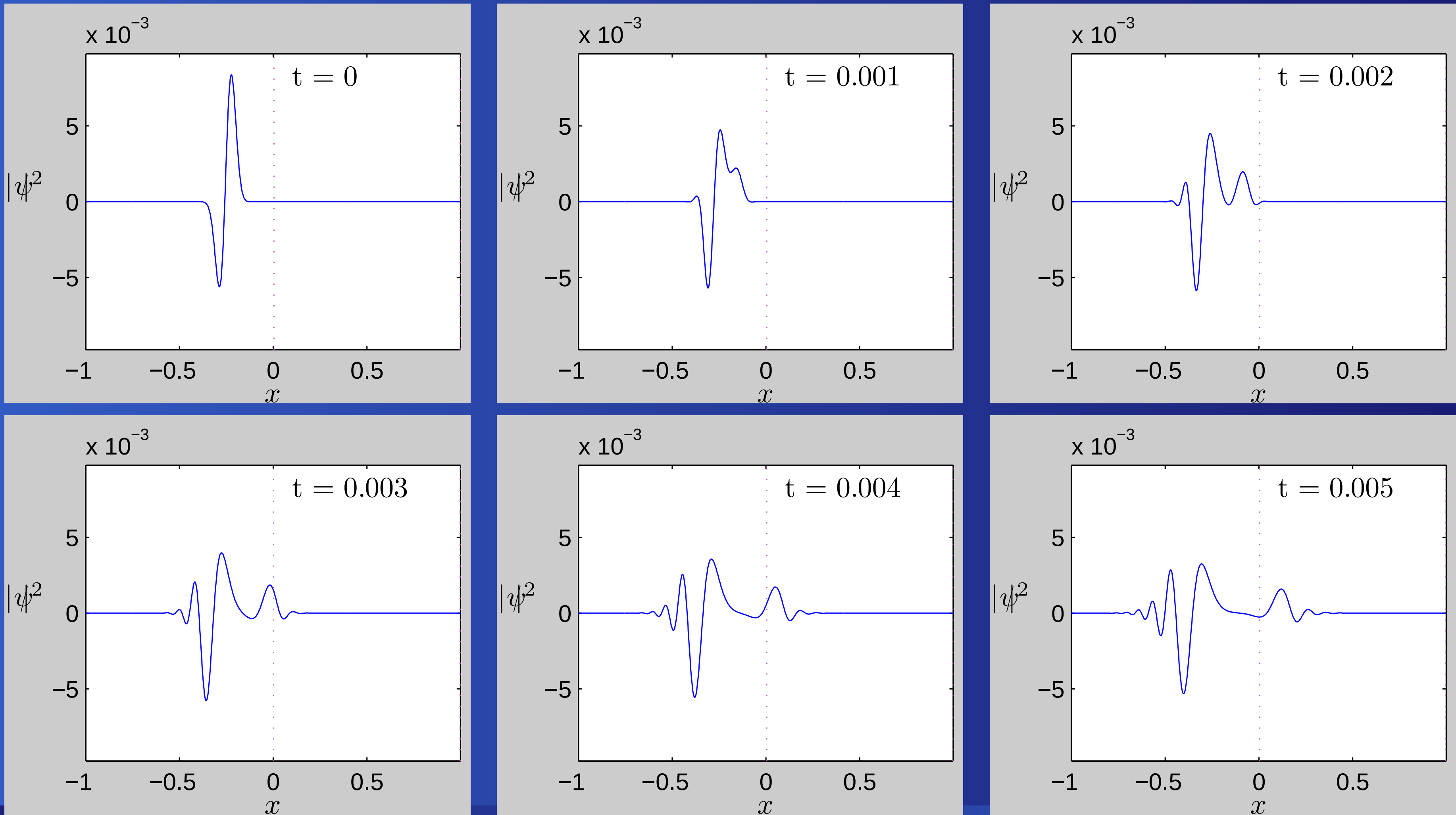
Gaussian wavepacket outside BH horizon

$x_{\text{center}} = -0.25,\ w = 10,\ V_0 = 200,\ C = 1974$



to be continued ...

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Acoustic BH configurations:

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Dimensionless units

computational units:

$$x_0 = L \qquad t_0 = mL^2/\hbar \qquad s_0 = \sqrt{1/x_0}$$

time dependent GP equation:

$$i\frac{\partial\overline{\psi}}{\partial\overline{t}} = -\frac{1}{2}\frac{d^2\overline{\psi}}{d\overline{x}^2} + \overline{V}(\overline{x})\overline{\psi} + C|\overline{\psi}|^2\overline{\psi}$$

time independent GP equation:

$$\overline{\mu}\,\overline{s} = -\frac{1}{2}\frac{d^2\overline{s}}{d\overline{x}^2} + \overline{V}(\overline{x})\overline{s} + C\overline{s}^3 + \frac{J}{\overline{s}^3}$$

- $C \equiv$ nonlinear interaction term
- $J \equiv$ current

velocity: $\overline{v} = \dfrac{\sqrt{2J}}{\overline{s}^2}$ speed of sound: $\overline{c} = \sqrt{C\overline{s}^2}$ phase: $\vartheta(\overline{x}) = \int \overline{v}\, d\overline{x}$

Note: henceforth we drop the bars for clarity