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Mott Insulator–BEC Transition and A Quantum Adiabatic Algorithm for Hilbert's Tenth Problem

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Aims of the talk

To point out that: the Mott insulator-BEC quantum phase transition is indeed a ***physical realisation*** of a quantum adiabatic algorithm for Hilbert's tenth problem for a class of Diophantine equations

Outline

- Quantum phase transition
- Mott insulator – BEC phase transition
- Bose-Hubbard model

- Quantum adiabatic computation
- An algorithm for Hilbert's tenth problem

- Intimate connection between the two for the class of linear Diophantine equations.

Quantum Phase Transition

- Different to thermal phase transition which is driven by the *thermal fluctuations* due to the competition between energy and entropy

- Driven by the *quantum fluctuations* (even at zero temperature) due to the competition between Hamiltonian terms which have different symmetry/conjugate properties

Mott insulator - BEC quantum phase transition

$$H = \int dx\, \psi^\dagger(x) \left(-\frac{\hbar^2}{2m} \nabla^2 + V_T(x) + V_L(x) \right) \psi(x) + \frac{g}{2} \int dx\, \psi^\dagger(x) \psi^\dagger(x) \psi(x) \psi(x)$$

where

$$g = 4\pi a_s \hbar^2 / m$$

$$V_T = \frac{m}{2} \left(\omega_x x^2 + \omega_y y^2 + \omega_z z^2 \right)$$

$$V_L = V_0 \left(\sin^2(k_x x) + \sin^2(k_y y) + \sin^2(k_z z) \right)$$

Wannier state in the n-th band

$$w_n(x - x_i) \sim \sum_q e^{-iqx_i} \phi_q^{(n)}(x)$$

$$\psi(x) = \sum_i a_i w_0(x - x_i)$$

Thus, in the lowest lying band

$$H = -\sum_{\langle ij \rangle} J_{ij}\, a_i^\dagger a_j + \sum_i \epsilon_i\, a_i^\dagger a_i + \frac{1}{2} \sum_{ijkl} U_{ijkl}\, a_i^\dagger a_j^\dagger a_k a_l$$

where

$$\epsilon_i = \int dx\, V_T(x) |w_0(x - x_i)|^2 \approx V_T(x_i)$$

$$J_{ij} = -\int dx\, w_0(x - x_i) \left(-\frac{\hbar^2}{2m} \nabla^2 + V_T(x) + V_L(x) \right) w_0(x - x_j)$$

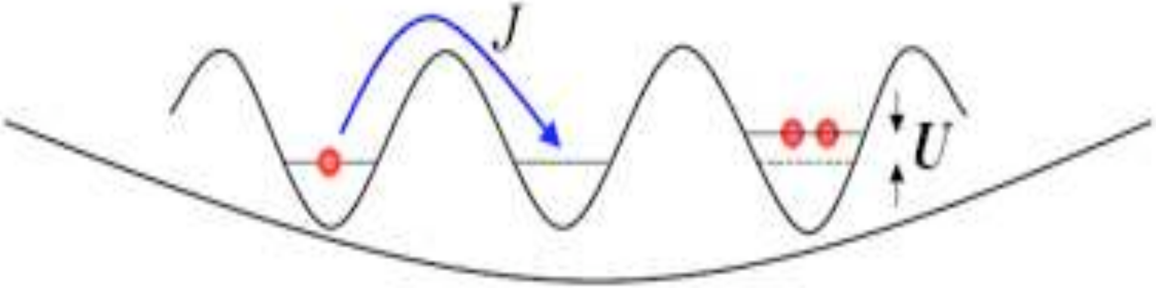
$$U_{ijkl} = g \int dx\, w_0(x - x_i) w_0(x - x_j) w_0(x - x_k) w_0(x - x_l)$$

We then have the Bose-Hubbard Model in some approximations:

$$H_{\mathrm{BH}} = -J \sum_{ij} a_i^\dagger a_j + \sum_i \epsilon_i n_i + \frac{1}{2} U \sum_i n_i (n_i - 1) - \sum_i \mu_i n_i$$

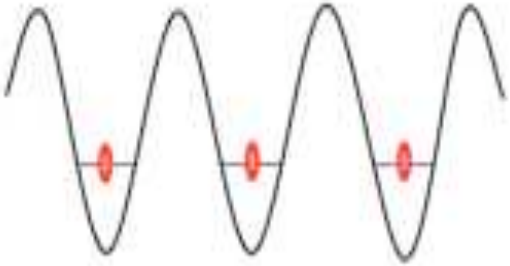
Bose-Hubbard Hamiltonian:

$$H = -J\sum_{\langle i,j\rangle}\hat{a}_i^{+}\hat{a}_j + \sum_i \varepsilon_i n_i + \tfrac{1}{2}U\sum_i \hat{n}_i\left(\hat{n}_i - 1\right)$$



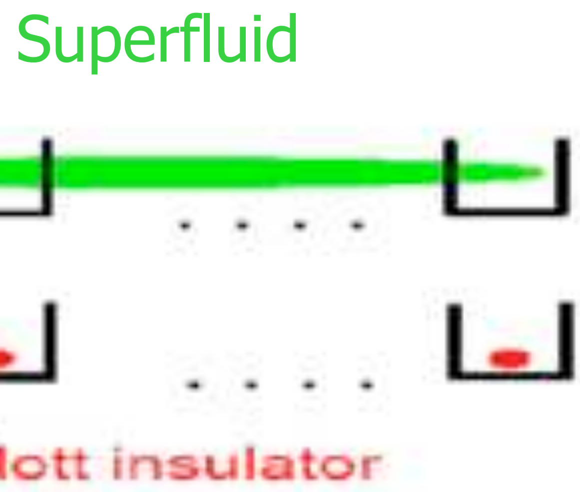
$U << J$: superfluid

$U >> J$: Mott insulator



Zoller

$$H_{\mathrm{SF}} = -J \sum_{\langle l,m \rangle} a_l^\dagger a_m \qquad |\Psi_{\mathrm{SF}}\rangle_{U=0} \propto \left(\sum_{i=1}^{M} \hat{a}_i^\dagger \right)^N |0\rangle$$



$$H_{\mathrm{MI}} = \frac{U}{2} \sum_l n_l (n_l - 1) \qquad |\Psi_{\mathrm{MI}}\rangle_{J=0} \propto \prod_{i=1}^{M} (\hat{a}_i^\dagger)^n |0\rangle$$

Zoller

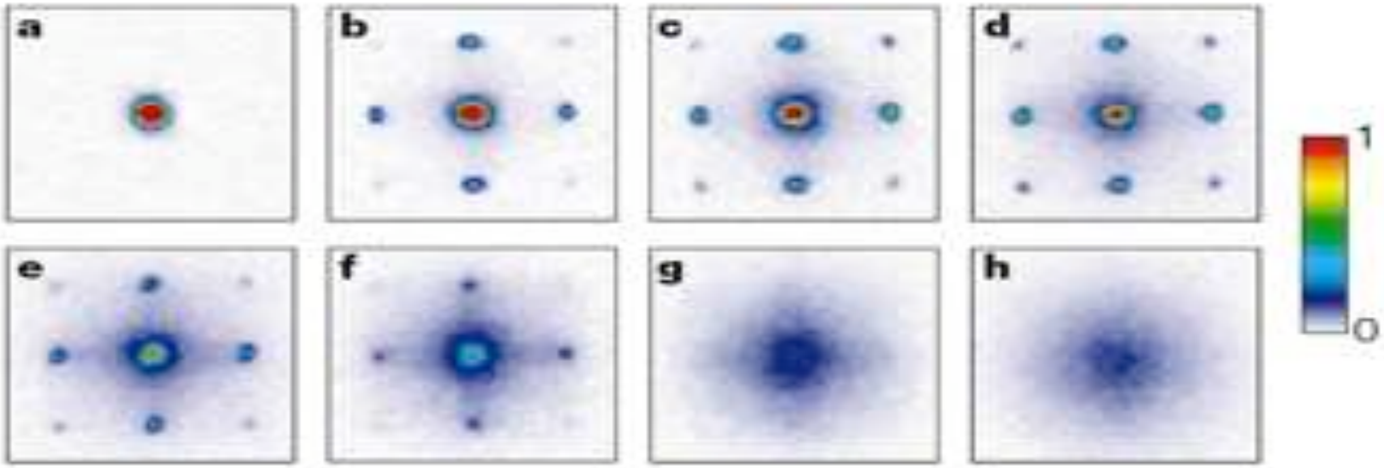
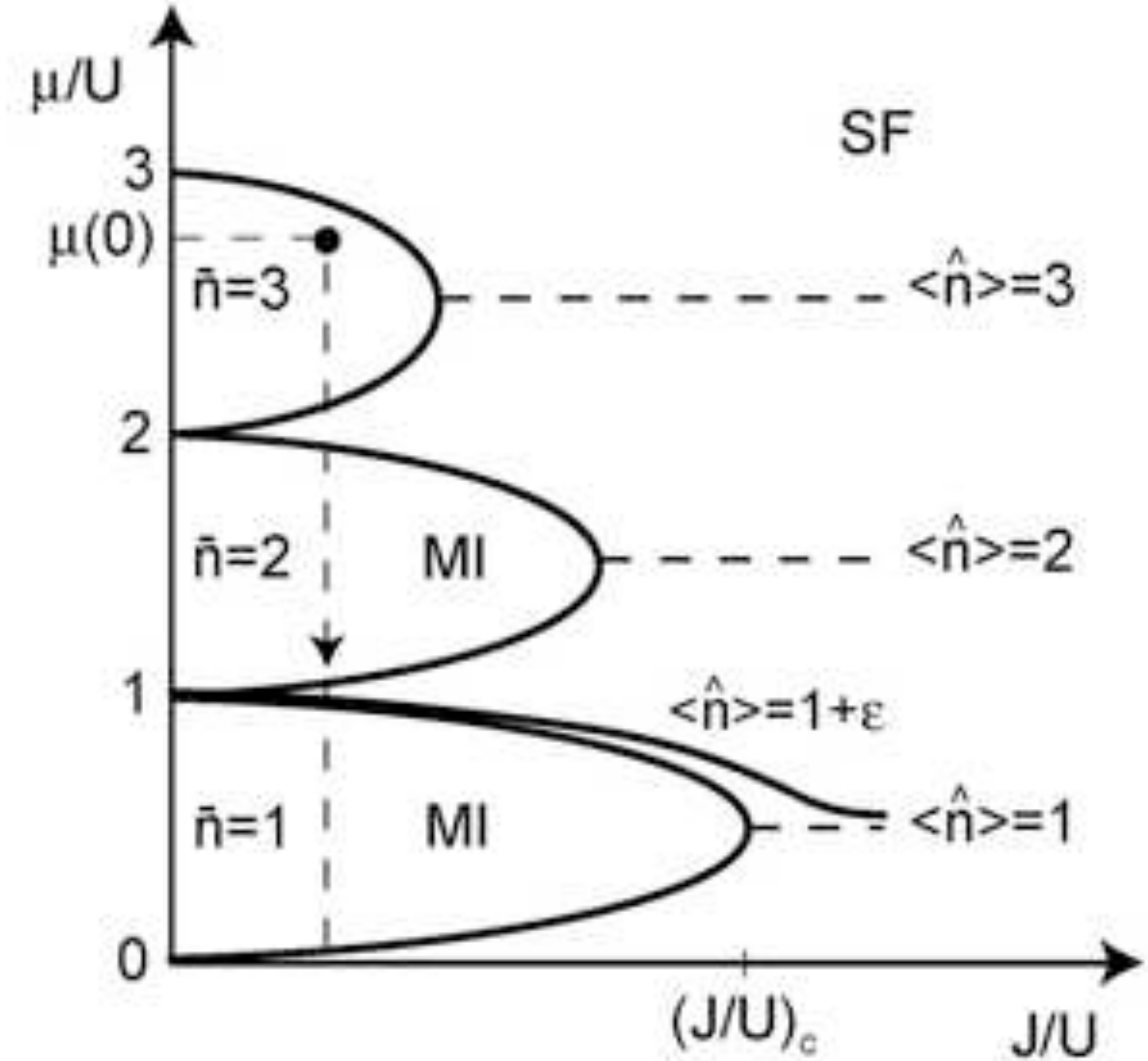


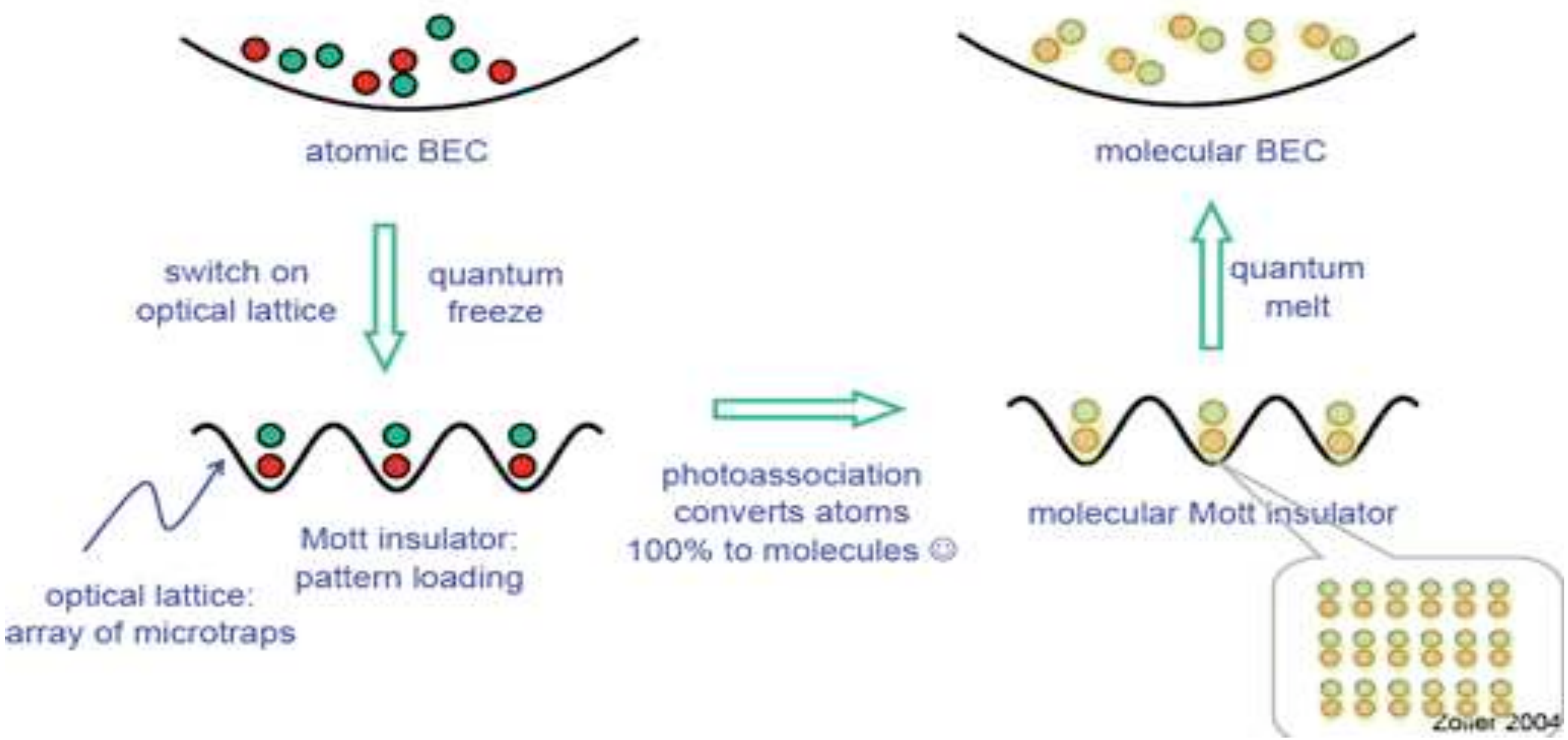
Figure 2 Absorption images of multiple matter wave interference patterns. These were obtained after suddenly releasing the atoms from an optical lattice potential with different potential depths V_0 after a time of flight of 15 ms. Values of V_0 were: **a**, 0 E_r; **b**, 3 E_r; **c**, 7 E_r; **d**, 10 E_r; **e**, 13 E_r; **f**, 14 E_r; **g**, 16 E_r; and **h**, 20 E_r.

M. Greiner *et al.*, Nature (London) **415**, 39 (2002)

W. Zwerger, J. Opt. **B5**, (2003) S9

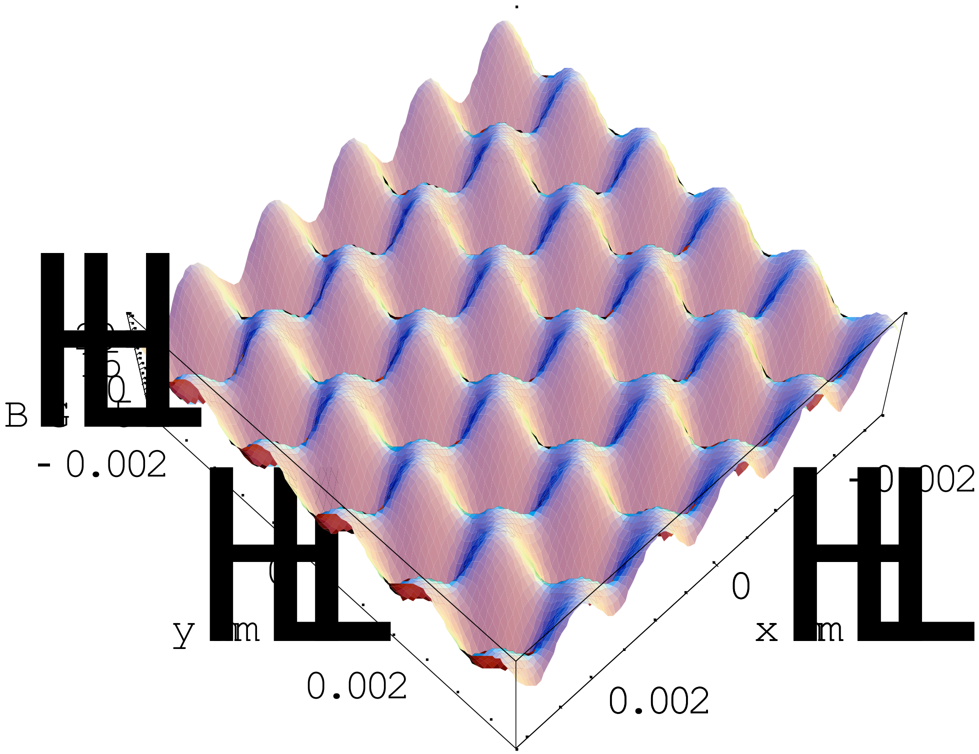


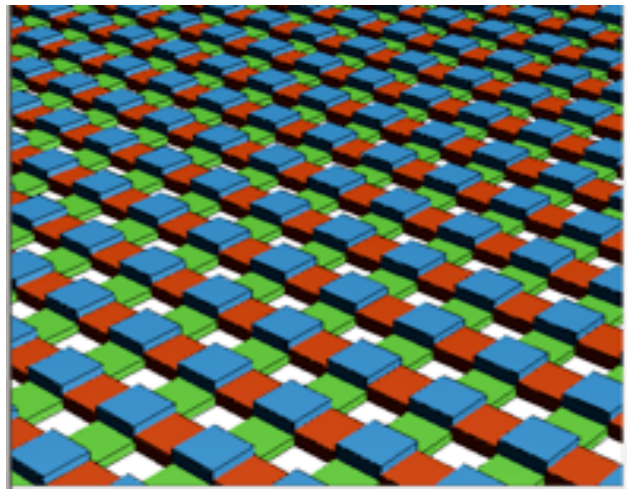
- molecular BEC via a quantum phase transition

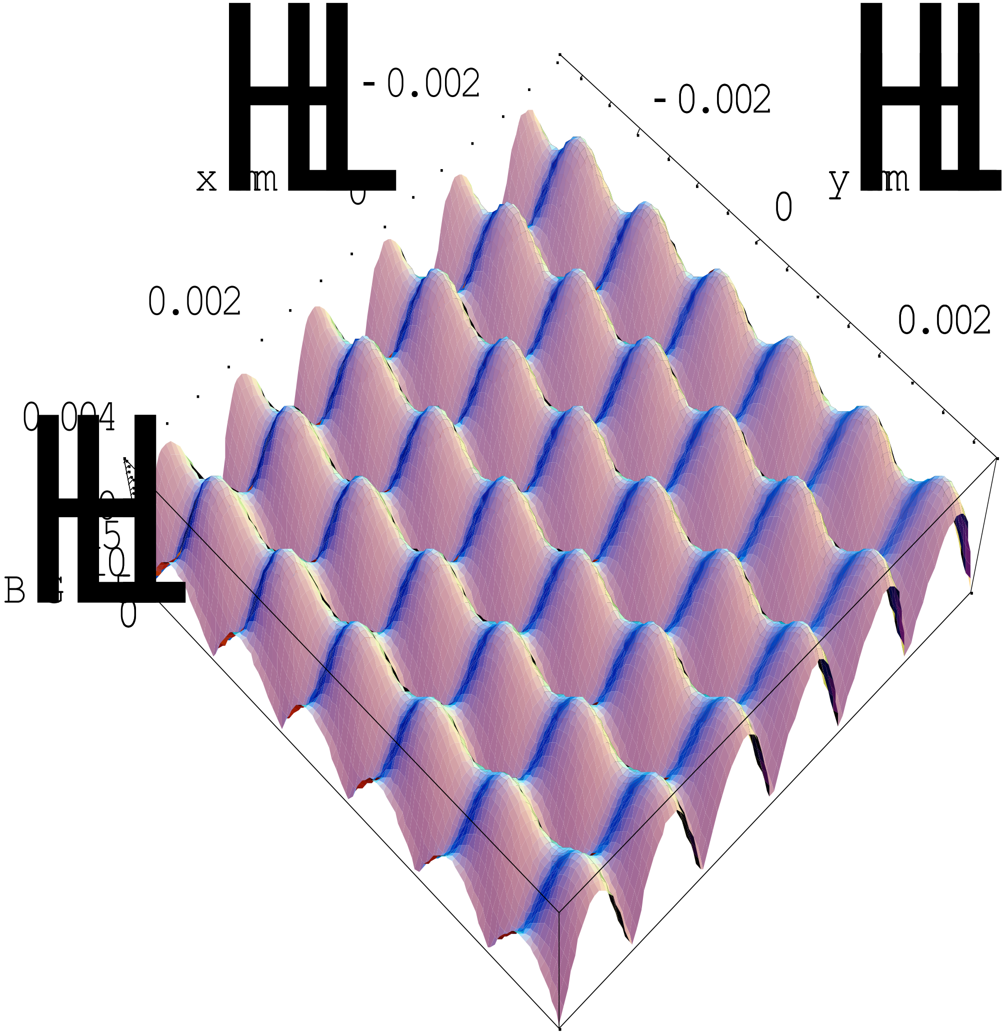


Magnetic Lattices

(courtesy Saeed Ghanbari)






-0.002
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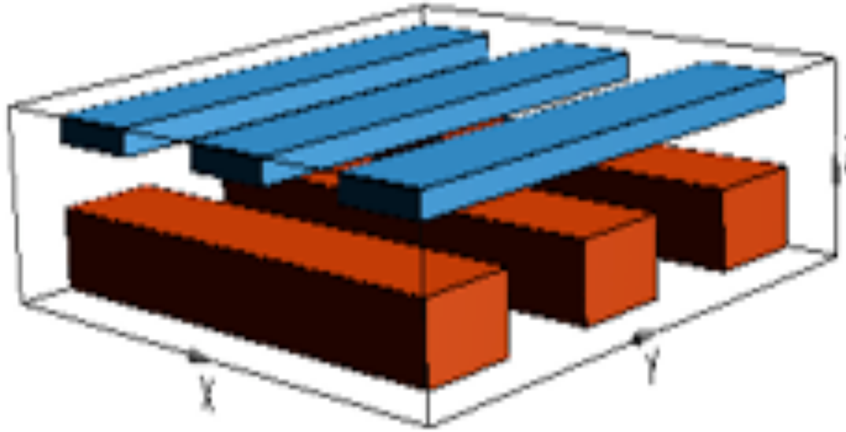


Other applications of the Bose-Hubbard model

- Arrays of Josephson junctions
- Granular and short-correlation-length superconductors
- Flux lattices in type-II superconductors
- Ultra-cold atoms in periodic lattices
- A route to molecular BEC
- A route to neutral-atom quantum computers
- NOW: ***Quantum adiabatic computation for Hilbert's tenth problem***

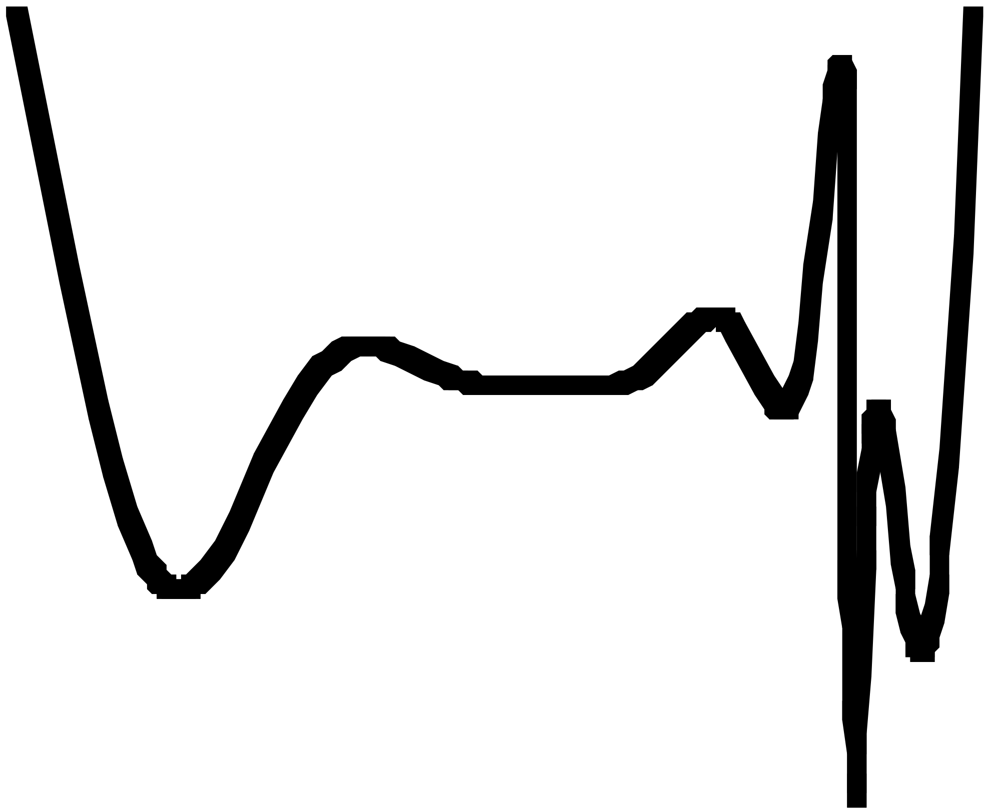
Farhi et al., quant-ph/0001106

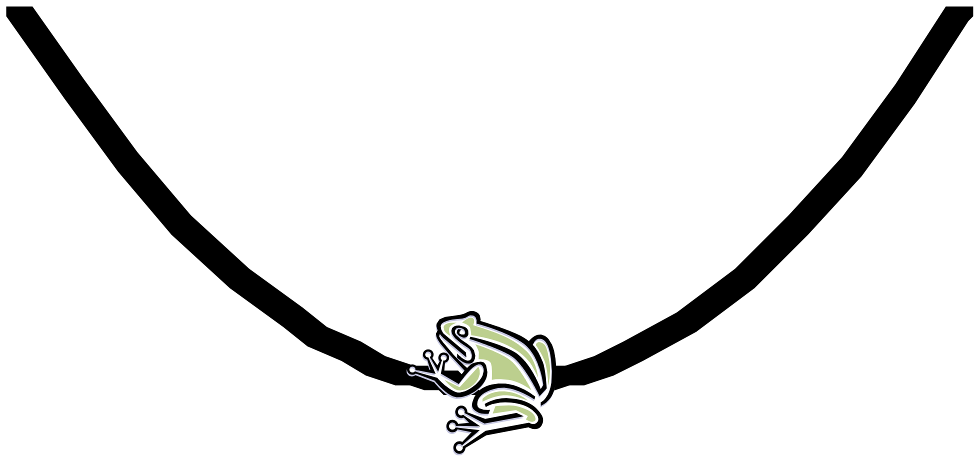
Quantum Adiabatic Computation

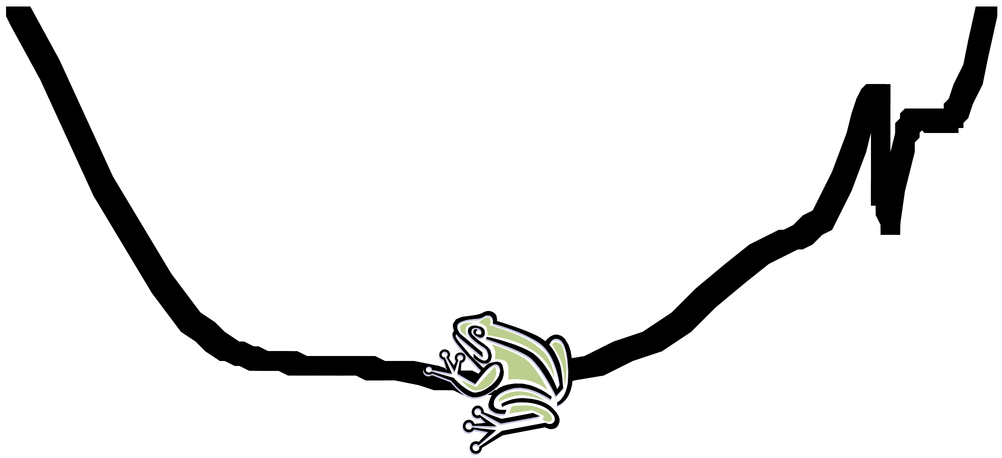
- Solution is encoded in the ground state |***g***> of some Hamiltonian H_P
- We then start with a readily available |***g'***> which is the ground state of another Hamiltonian H_I
- We then interpolate between the Hamiltonians in the time T

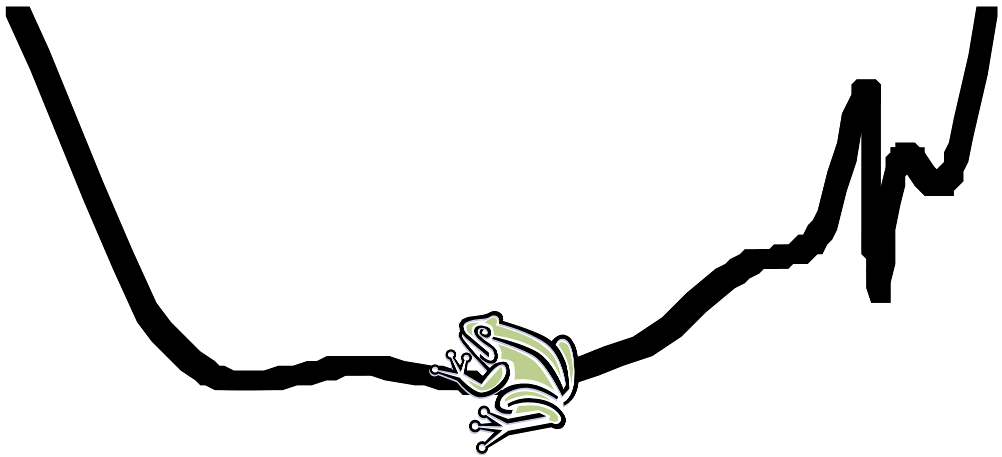
$$H(t) = (1 - t/T)\, H_I + (t/T)\, H_P$$

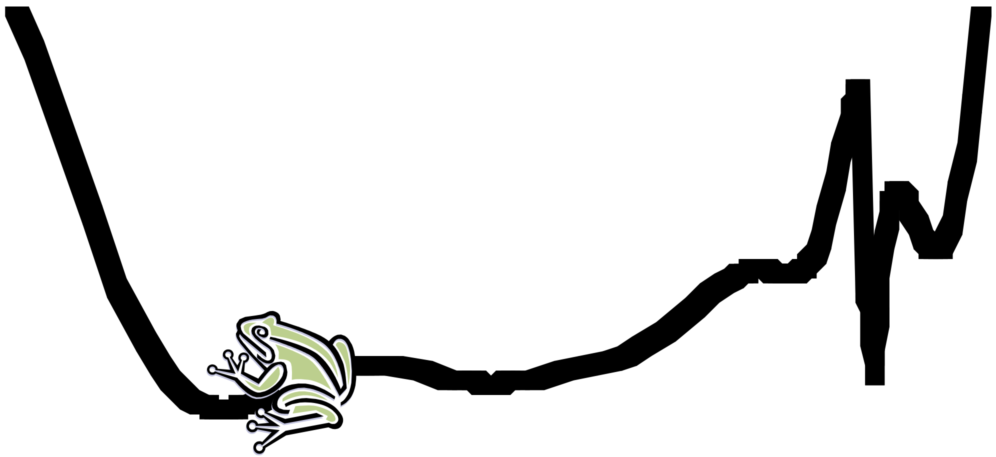
- If the ***quantum adiabatic theorem*** is satisfied, we can then obtain |***g***> from |***g'***> , and thus the solution!!!

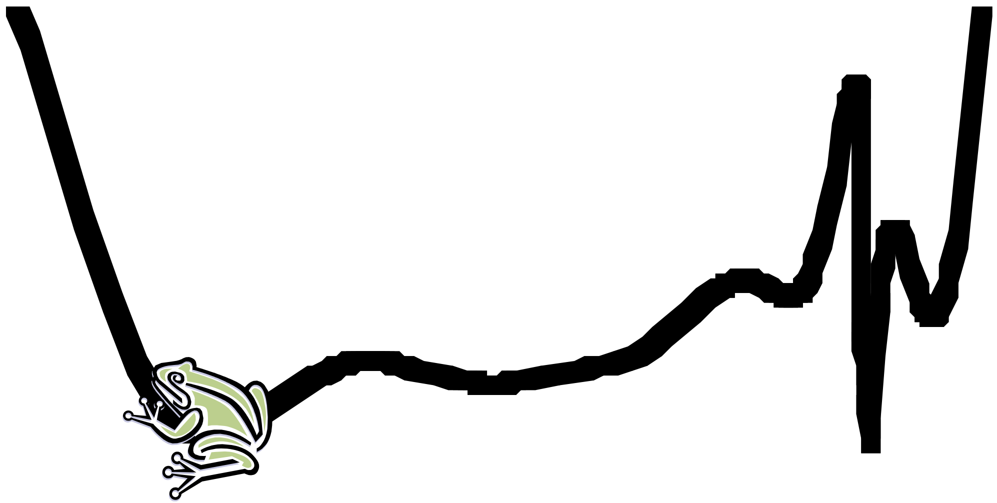


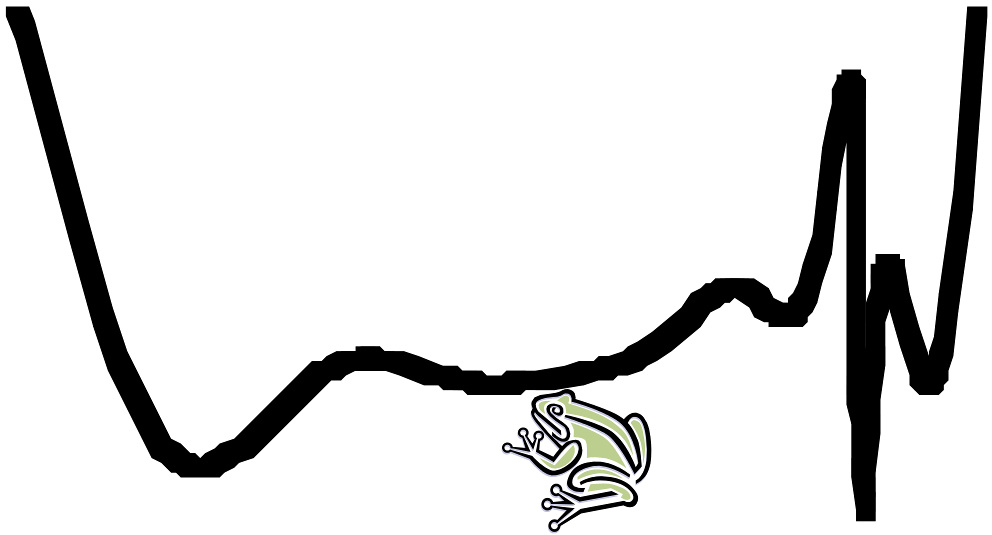


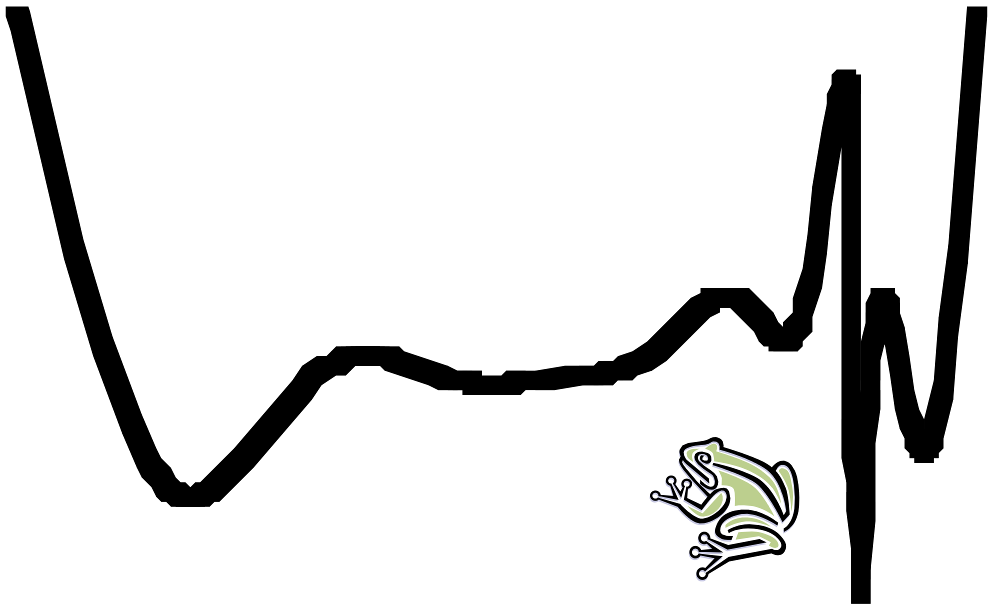


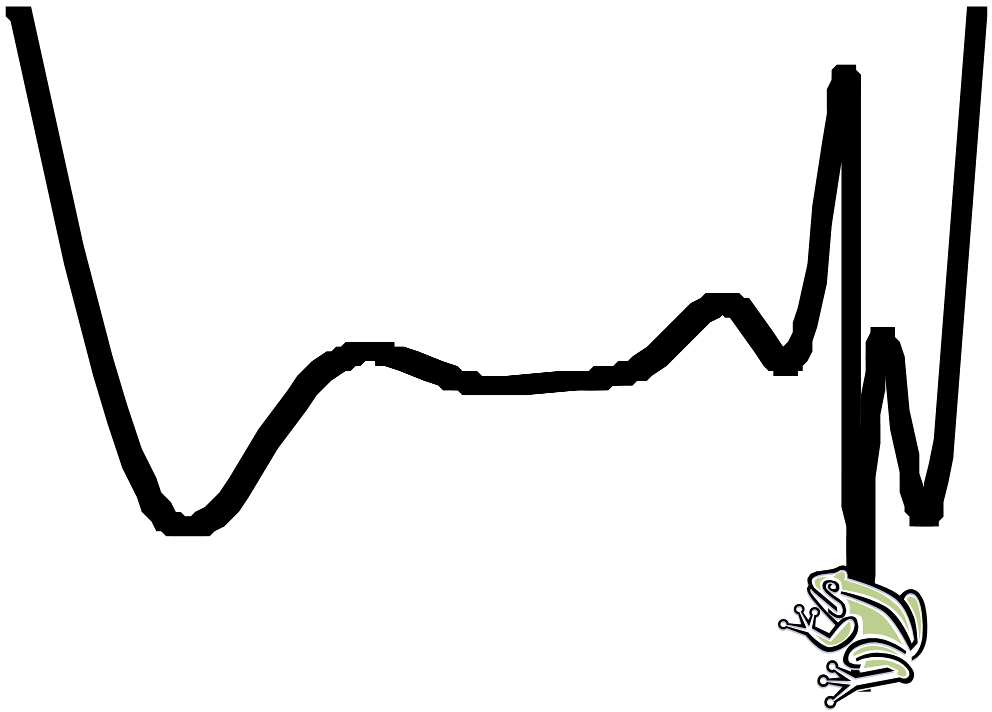


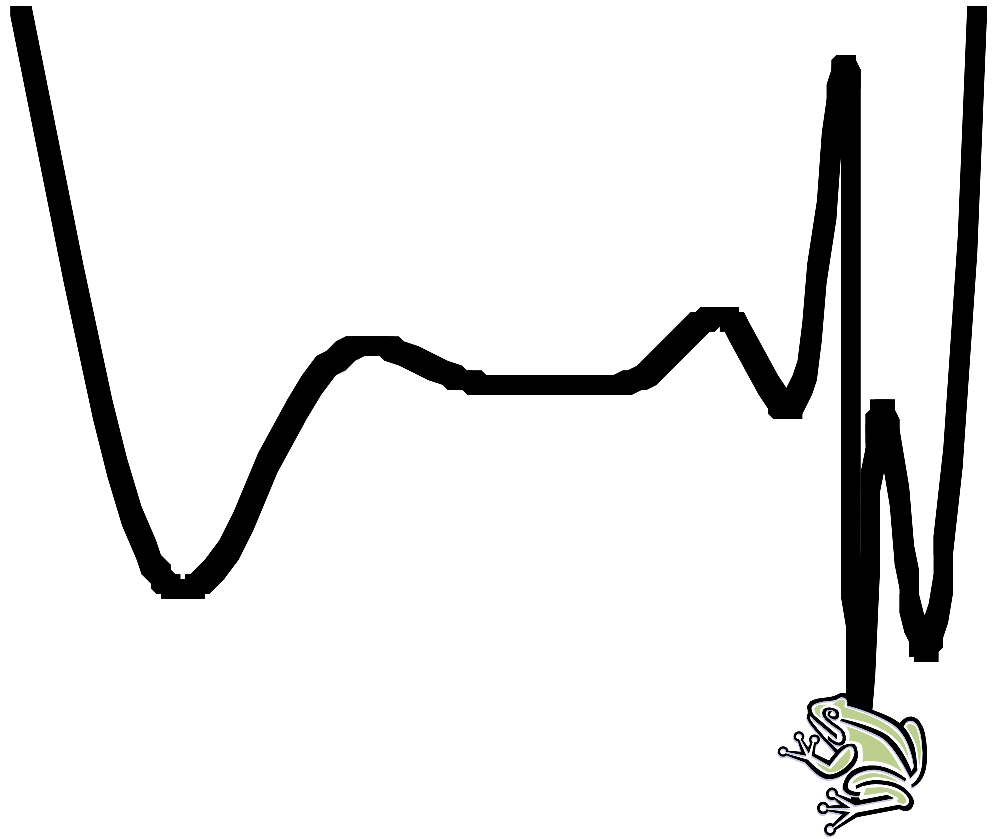












Fermat's last theorem

$$(x+1)^3+(y+1)^3-(z+1)^3 \stackrel{?}{=} 0$$

??What about

$$(x+1)^3+(y+1)^3-(z+1)^3+xyz \stackrel{?}{=} 0$$

What about Goldbach's conjecture?

Distribution of zeroes of Riemann Zeta functions?

Hilbert's tenth problem (1900)

Given any polynomial equation with any number of unknowns and with integer coefficients: To devise a universal process according to which it can be determined by a finite number of operations whether the equation has integer solutions.

Turing halting problem (1937)

A Turing computation is equivalent to the computation of a partial recursive function, which is only defined for a subset of the integers; as this domain is classically undecidable, one cannot always tell in advance whether the Turing machine will halt (that is, whether the input is in the domain of the partial recursive function) or not (when the input is not in the domain).

$$\text{Turing halting problem} \overset{\text{Matiyasevich(1970)}}{\equiv} \text{Hilbert's tenth problem}$$

An observation

It suffices to consider nonnegative solutions of a Diophantine equation. Let us consider the example:

$$(x+1)^3 + (y+1)^3 - (z+1)^3 + cxyz = 0, \qquad c \in Z.$$

We construct the following Hamiltonian:

$$H_P = \left((a_x^\dagger a_x + 1)^3 + (a_y^\dagger a_y + 1)^3 - (a_z^\dagger a_z + 1)^3 + c(a_x^\dagger a_x)(a_y^\dagger a_y)(a_z^\dagger a_z)\right)^2,$$

and find its ground state

$$N_j|g\rangle = n_j|g\rangle,$$
$$H_P|g\rangle = \left((n_x+1)^3 + (n_y+1)^3 - (n_z+1)^3 + cn_xn_yn_z\right)^2|g\rangle \equiv E_g|g\rangle.$$

Thus a projective measurement of the energy E_g of the ground state $|g\rangle$ will yield the answer for the decision problem: **The Diophantine equation has at least one integer solution if and only if $E_g = 0$, and has not otherwise.**

(If $c = 0$ in our example, we know that $E_g > 0$ from the Fermat's last theorem.)

Mott insulator - BEC transition and linear Diophantine equations

Quantum phase transition from the (bosonic) Hubbard model:

$$H = -J(t)\sum_{ij}(a_j^\dagger a_i + a_i^\dagger a_j) + \frac{U(t)}{2}\sum_i n_i(n_i - c_i),$$

$$c_i = 1 - 2(\epsilon_i - \mu_i)/U$$

- $J \gg U$: $|g'\rangle \sim \left(\frac{1}{\sqrt{M}}\sum_i^M a_i^\dagger\right)^N |0\rangle$ is approximately the coherent state in each well i, M and $N \gg 1$
- $J \ll U$: $|g\rangle \sim \prod_i^M (a_i^\dagger)^d |0\rangle$, for some d

On the other hand, linear Diophantine equations of the form $x - d = 0$ leads to

$$\mathcal{H}(t) = \left(1 - \frac{t}{T}\right) H_I + \frac{t}{T} H_P$$

$$H_P = (n - d)^2 = n(n - 2d) + d^2$$

Thus, in each well i,

$$d = \frac{c_i}{2} = \frac{1}{2} - \frac{\epsilon_i - \mu_i}{U}$$

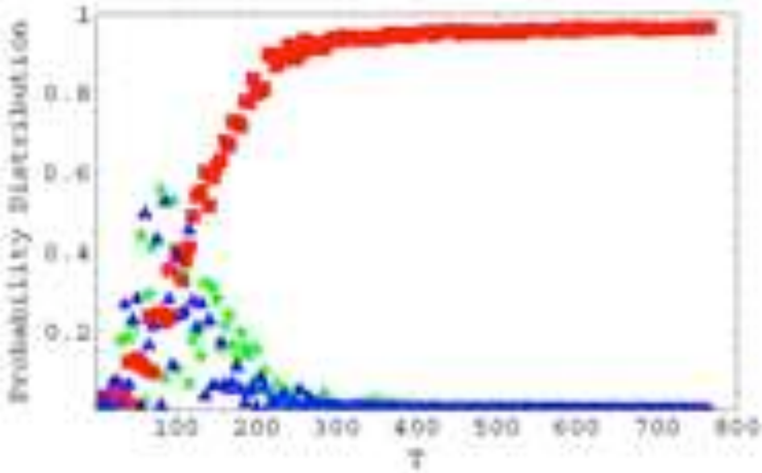


Figure 1: Equation $X - 20 = 0$ with $\alpha = 2.0$. The initial Fock space has only up to $|n\rangle = |14\rangle$; but the final ground state $|20\rangle$, in red, can be reached and identified. Green and blue are $|19\rangle$ and $|21\rangle$.

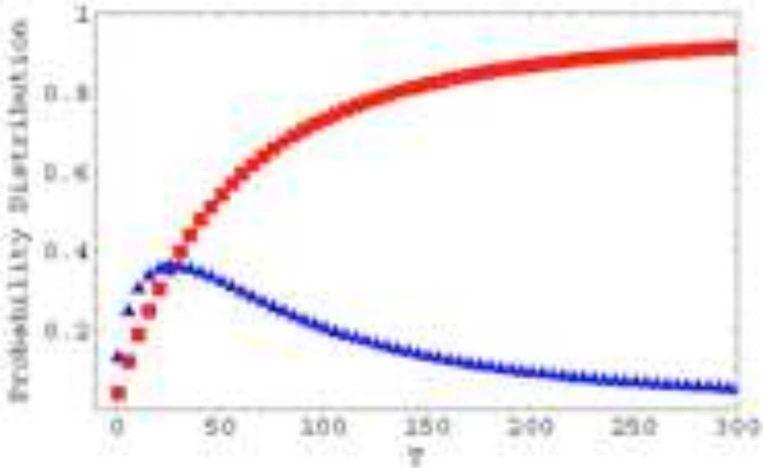


Figure 4: Equation $X + 20 = 0$, which has no positive solution. The red dots are for the ground state $|0\rangle$; blue dots $|1\rangle$. Initial size of the trucnated Fock space is $m_x = 8$.

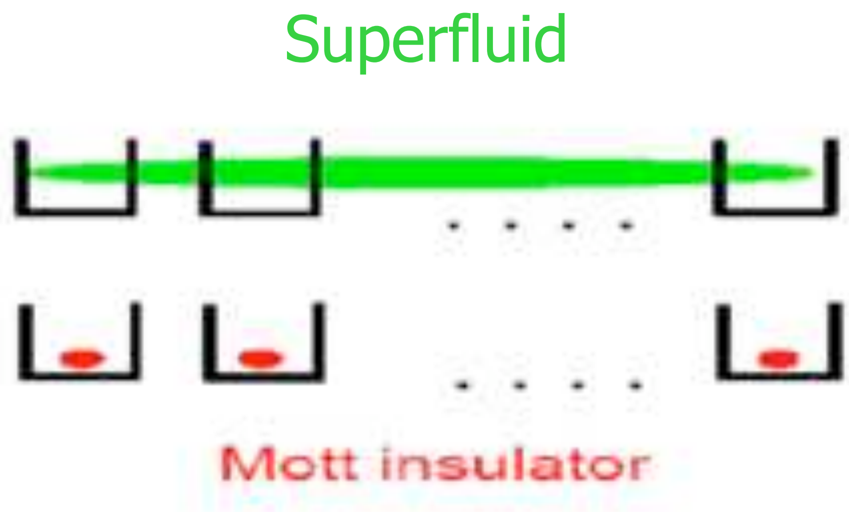
Can be genralised to the class of linear Diophantine equations with more than one variable:

$$ax + by + cz + d = 0$$

All in the quantum adiabatic process!

- *Why the equivalence?* the quantum phase transition is after all ***a quantum adiabatic process,*** as much as the quantum adiabatic computation

$$H_{\mathrm{SF}} = -J \sum_{\langle l,m \rangle} a_l^{\dagger} a_m \qquad |\Psi_{\mathrm{SF}}\rangle_{U=0} \propto \left(\sum_{i=1}^{M} \hat{a}_i^{\dagger} \right)^N |0\rangle$$



$$H_{\mathrm{MI}} = \frac{U}{2} \sum_{l} n_l (n_l - 1) \qquad |\Psi_{\mathrm{MI}}\rangle_{J=0} \propto \prod_{i=1}^{M} (\hat{a}_i^{\dagger})^n |0\rangle$$

All in the quantum adiabatic process!

- *Why the equivalence?* the quantum phase transition is after all ***a quantum adiabatic process,*** as much as the quantum adiabatic computation

- *Any gain?*
  - Quantum phase transition helps ***computing the noncomputable*** (other classes of Diophantine equations)
  - A result borrowed from the quantum adiabatic algorithm helps us to ***identify the onset*** of the quantum phase transition

- Published:
  - *Minds and Machine* 12 (2002) 541- 612.
  - *Contemporary Physics* 44 (2003) 51- 71
  - *Int. J. Theo. Phys.* 42 (2003) 1461 – 1478
  - (With T. Ord) *Fundamenta Informaticae* 56 (2003) 273 – 284
  - *Proc. Roy. Soc.* A 460 (2004) 1535-
  - *Theoretical Computer Science* 317 (2004) 93-104
  - (With T. Ord) to appear in *British Journal for Philosophy of Science*, math.LO/0307020
  - to appear in *International Journal of Quantum Information*, quant-ph/0407090
  - in Proceedings of SPIE Vol. 5105 *Quantum Information and Computation*, edited by Eric Donkor, Andrew R. Pirich, Howard E. Brandt, (SPIE, Bellingham, WA, 2003), pp. 89-95

- Unpublished
  - quant-ph/0310052: "Quantum adiabatic algorithm for Hilbert's tenth problem: I. The algorithm."
  - (With T. Ord) cs.OH/0401019: "Using biased coins as oracles."
  - quant-ph/0111062: "Gödel's Incompleteness, Chaitin's W number and Quantum Physics."
  - quant-ph/0111020: "Reply to 'The quantum algorithm of Kieu does not solve the Hilbert's tenth problem'."
  - quant-ph/0403045: "Finiteness of the universe and computation beyond Turing computability."