Contents

Lectures 7-9

Bipolar junction transistor (BJT)

Simplest transistor model

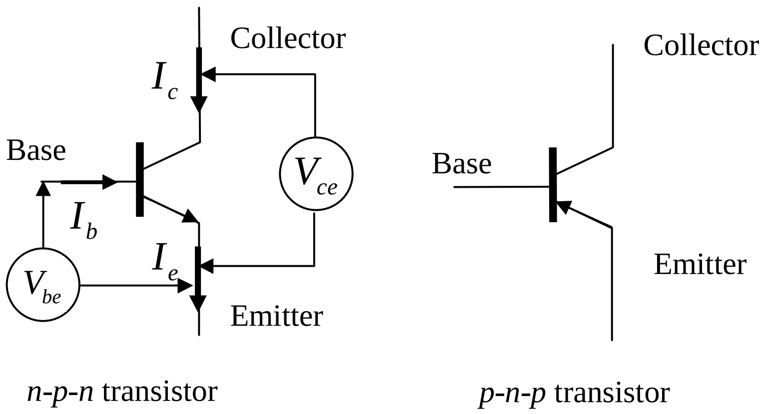


Fig. 40. BJT transistors. The arrow shows the direction of the emitter-collector and base-emitter currents

The following rules are used in the analysis of circuits based on *n-p-n* transistor (for p-n-p transistor change the polarities of all voltages and the directions of all currents).

1. The collector should be more positive than the emitter
2. Normally the base is more positive than the emitter but more negative than the collector. In this case, the base is $0.6 \div 0.8\ \text{V}$ more positive than the emitter (like in the case of a simple diode). Every transistor has limits for I_c, I_b, V_{ce} (specified for every type of transistors) which when exceeded damage the transistor.
3. The collector current is roughly proportional to the base current: $I_c = \beta I_b$. The current gain β is typically 100.

A transistor is a non linear device. For example, the collector's current nonlinearly depends on the base-emitter voltage. A simple model predicts that

$$I_c = I_s\left[\exp\left(\frac{V_{be}}{k_B T / q_e}\right) - 1\right] \approx I_s \exp\left(\frac{V_{be}}{k_B T / q_e}\right), \tag{1}$$

where $k_B T / q_e = 25.3\ \text{mV}$ at room temperature and I_S is a saturation current.

Also the base current amplification β depends on the base current. Therefore a feedback principle is widely used in transistor based circuits. This principle has already been discussed in connection with operational amplifiers. Here we will treat it a bit more generally and with more details.

Feedback theory

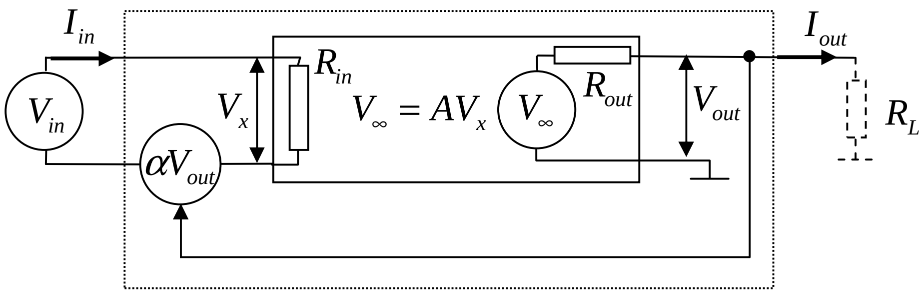


Fig. 41. Series feedback in an amplifier

An amplifier is shown in Fig. **41**. The open circuit voltage V_∞ (that is the voltage when there is no load resistor connected to the output) is proportional to the voltage between input terminals of the amplifier V_x. A fraction of the output voltage is fed back in to the input in series with the external input voltage V_{in}. When $\alpha A < 0$ the feedback is called "negative" . When $\alpha A > 0$, the feedback is "positive".

First we assume that $R_L = \infty$. In this case $V_{out} = V_\infty$ and the following equalities hold

$$\begin{cases} V_x = V_{in} + \alpha V_{out} \\ V_{out} = AV_x \end{cases} \tag{2}$$

We substitute the upper equation into the lower one to obtain

$$V_{out} = A\left(V_{in} + \alpha V_{out}\right) \tag{3}$$

which can be solved for V_{out}.

$$V_{out} = \frac{A}{1-\alpha A} V_{in} \tag{4}$$

The actual input voltage V_x can be obtained from the first equation of the coupled equations (2)

$$V_x = V_{in}\left(1+\frac{\alpha A}{1-\alpha A}\right) = V_{in}\frac{1}{1-\alpha A}$$

Calculation of the effective input resistance

The input resistance (impedance) is defined as the ratio of the input voltage to the input current. For an amplifier with a feedback (a device inside the dotted box in Fig. **41**) one gets

$$R'_{in} \equiv \frac{V_{in}}{I_{in}} = \frac{V_x\left(1-\alpha A\right)}{I_{in}} \tag{5}$$

Because $I_{in} = V_x / R_{in}$, the final result reads

$$R'_{in} = R_{in}\left(1-\alpha A\right) \tag{6}$$

Calculation of the effective output resistance

The output resistance of the device in the dotted box in Fig. **41** is the ratio of the open circuit output voltage to the short-circuit (that is when $R_L \to 0$) output current. In the short-circuit regime, $V_{out} = 0$ and therefore the short-circuit current $I_{sc} = V_\infty / R_L = AV_{in} / R_L$ (the second equality follows because $V_x = V_{in}$ in this case). The open circuit voltage $V_{oc} = \dfrac{AV_{in}}{1-\alpha A}$ has been already calculated and the effective output resistance reads

$$R'_{out} = \frac{V_{oc}}{I_{sc}} = \frac{R_{out}}{1-\alpha A} \qquad (7)$$

Summary of important results for a series feedback.

1. If $\alpha A \to 1$ (apparently this is possible only with a positive feedback) then $V_{out} \to \infty$ and $V_x \to \infty$ (unless $V_{in} = 0$). The amplifier is unstable.
2. The effective input resistance is always larger than R_{in} if the feedback is negative.
3. The effective output resistance is smaller than R_{out} if the feedback is negative.
4. If $|A|$ is large, then the effective amplification depends only on the feedback parameter α.

Emitter follower.

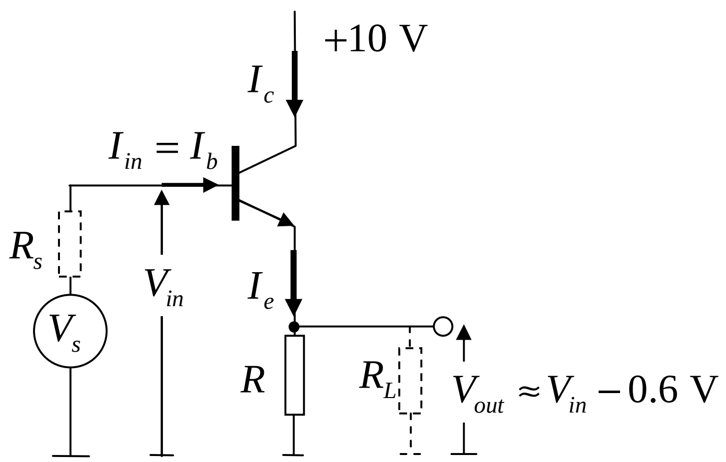


Fig. 42. Emitter follower.

Input impedance of emitter followers can be obtained using the general theory of a feedback (note that the voltage between the base and the emitter is the difference between the input voltage and the output voltage and therefore the feedback parameter $\alpha = -1$) but it is not difficult to calculate it from the "first principles".

$$\Delta I_e = \Delta I_b + \Delta I_c = \Delta I_b + \beta \Delta I_b = \Delta I_b \left(1+\beta\right)$$

$$\Delta V_{in} = \Delta V_{out}$$

$$R_{in} \equiv \frac{\Delta V_{in}}{\Delta I_{in}} = \frac{\Delta V_b}{\Delta I_b} = \frac{\Delta V_{out}}{\Delta I_e / \left(1+\beta\right)} = R\left(1+\beta\right)$$

An emitter follower allows using a voltage source which has a large internal resistance without a substantial drop of the voltage across the load. Calculations of the output resistance of an emitter follower are slightly more involved. For briefness, we introduce $\left(R \,||\, R_L\right)$ as a notation for the resistance of parallel resistors R_L and R

Kirchhoff's voltage law: $V_s - R_s \dfrac{I_e}{\beta+1} - I_e\left[R \,||\, R_L\right] = 0$

Solving this for I_e obtain $I_e = \dfrac{V_s\left(\beta+1\right)}{R_s + \left(\beta+1\right)\left[R \,||\, R_L\right]}$

The emitter voltage reads $V_e = I_e\left[R \,||\, R_L\right] = \dfrac{V_s\left(\beta+1\right)\left[R \,||\, R_L\right]}{R_s + \left(\beta+1\right)\left[R \,||\, R_L\right]}$

The output resistance is

$$R_{out} = \frac{\Delta V_{e,R_L=\infty}}{\Delta I_{e,R_L=0}} = \frac{\dfrac{V_s\left(\beta+1\right)R}{R_s+\left(\beta+1\right)R}}{\dfrac{V_s\left(\beta+1\right)}{R_s}} = \frac{RR_s}{R_s + \left(\beta+1\right)R} \approx \frac{R_s}{\beta+1}.$$

The emitter follower transforms the internal resistance of the voltage source to a much lower value.

What is an operating point of a transistor?

The emitter follower shown in Fig. **42** can work properly only if a positive potential (relative to the ground) is applied to the base. To lift this limitation, a proper operation point of the transistor should be set.

The operation point of a transistor (or a Q-point) is characterized by the values of DC voltages across its terminals (base-emitter and collector-emitter voltages) and the corresponding DC transistor currents (emitter and collector currents). In the absence of the AC signal, the transistors' base should have a higher potential (an *npn*-transistor is assumed in all the examples in this chapter) than the emitter's potential. This ensures that the transistor operates properly (the base stays at a higher potential relative to the emitter) when a small AC input signal is applied (through a capacitor). The two resistors R_1 and R_2 divide the 10 V of the power supply. The base potential is then given by

$$V_b = \frac{V_{CC} R_2}{R_1 + R_2}$$

The emitter potential follows from the Ohm's law and is

$$V_e = I_e R_e$$

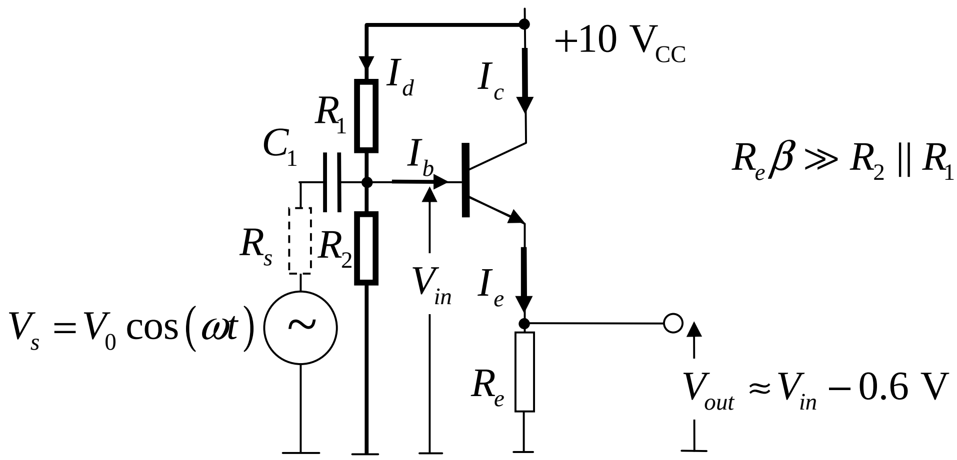
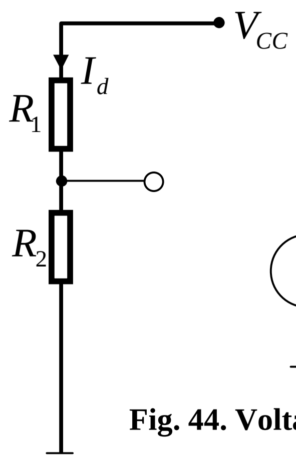


Fig. 43. Biased emitter follower.

The emitter follower does not overload the voltage divider if the equivalent resistance of the biased voltage source is much smaller than the equivalent input resistance of the emitter follower.



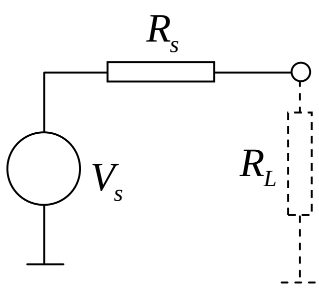


Fig. 44. Voltage divider

The equivalent voltage source for the circuit shown in Fig. 36. The values are

$$V_s = V_{CC}\frac{R_2}{R_1+R_2}$$

$$R_s = \frac{V_s}{I_{R_L=0}} = \frac{V_s}{V_{CC}/R_1} = \frac{R_1R_2}{R_1+R_2} = R_1 \parallel R_2$$

Therefore $R_e\beta \gg R_2 \parallel R_1$. Usually $R_e\beta \approx 10\cdot R_2 \parallel R_1$.

The choice of capacitor C_1 depends on the lowest frequency at which the follower is expected to work. For the ac input signal the three resistors R_1, R_2, and βR_e are connected in parallel and they form a high pass filter together with C_1. The amplitude attenuation of this filter is

$$\frac{1}{\sqrt{1+\left(\omega\left[R_1 \parallel R_2 \parallel \beta R_e\right]C_1\right)^2}}$$

and equals 3 dB when $\omega\left[R_1 \parallel R_2 \parallel \beta R_e\right]C_1 \equiv 2\pi f\left[R_1 \parallel R_2 \parallel \beta R_e\right]C_1 = 1$.

How to design an emitter follower.

1. Select V_{CC}
2. Select I_e. Typically, $I_e = 1$ mA for a low power transistor.
3. Select the lowest frequency $f_{\min}$ at which the follower should operate.
4. Select the value of β. Typically, $\beta \approx 100 \div 300$.
5. Select the operating point of the voltage emitter and calculate the value of $R_e = 0.5V_{CC}/I_e$.
6. Calculate the operation point of the base voltage $V_b = 0.5V_{CC} + 0.6$ V.
7. Calculate $V_{R1} = V_{CC} - \left(0.5V_{CC} + 0.6\ \text{V}\right) = 0.5V_{CC} - 0.6$ V, the voltage across R_1 and the ratio $R_1/R_2 = V_{R1}/V_b$.
8. Calculate the values of R_1 and R_2 by selecting the relation $R_e\beta \approx 10R_2 \parallel R_1$.

9. Calculate the value of the capacitance using the relation
   $2\pi f_{\min}\left[R_1 \,||\, R_2 \,||\, \beta R_e\right]C_1 = 1$

Common emitter amplifier

Emitter follower is not actually a voltage amplifier but a current amplifier (it transforms a small base current into a mach larger emitter current).

A common emitter amplifier is an amplifier where the output voltage can be larger than the input voltage.

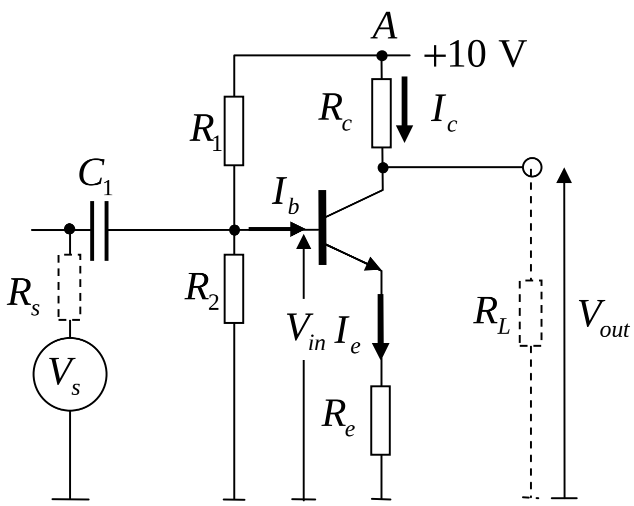


Fig. 45. Common emitter amplifier.

Because we consider all circuits in a linear approximation, DC operation and AC operation can be analysed separately.

Notice that R_e provides a negative feedback for DC and AC currents. Think of the DC collector current and its effect on the output voltage and on the base-emitter voltage. If the collector current increases, the emitter voltage also increases and therefore the base-emitter voltage decreases leading to a smaller collector current. The negative feedback in the common emitter amplifier makes the gain (amplification) almost independent on the current gain β of the transistor. The analysis of the common emitter amplifier is based on the two Kirchhoff's laws and Ohm's laws. The AC part of the input voltage ΔV_{in} is applied to the base of the transistor. Using the Kirchhoff's voltage law we get

$$\Delta V_{in} = \Delta V_b = \Delta I_e R_e + \Delta V_{be} = \Delta I_e R_e + \Delta I_c r_e \approx \Delta I_e R_e \,,$$

where we have neglected the *intrinsic emitter resistance* r_e. The *intrinsic emitter resistance* of the transistor is

$$r_e \equiv \frac{\Delta V_{be}}{\Delta I_c} = \frac{k_B T}{q}\frac{1}{I_c} = \frac{25\text{ mV}}{I_c}$$

That is if the collector current is 1 mA, the intrinsic resistance is only 25 Ω and can be neglected in comparison to the value of R_e if $r_e \ll R_e$.

Note, the equation for r_e is derived from

$$\Delta I_c = \frac{q}{k_B T} I_s \exp\left(\frac{V_{be}}{k_B T / q_e}\right)\Delta V_{be} = \frac{q}{k_B T} I_c \Delta V_{be} \rightarrow \frac{\Delta V_{be}}{\Delta I_c} = \frac{k_B T}{q I_c}$$

and therefore the value of r_e is also temperature dependent. According to the Ohm's law, the output voltage on the resistor R_c is

$$\Delta V_{out} = \Delta I_c R_c \approx \Delta I_e R_c,$$

where approximation is made using the fact that $I_e = I_c + I_b$ and that the base current is much smaller than the collector current.

$$A = \frac{\Delta V_{out}}{\Delta V_{in}} \approx \frac{R_c}{R_e}$$

To analyse of the input resistance, note that for alternating potentials, point A and the ground are connected (the difference between A and the ground has only a DC component). Therefore R_1 and R_2 are connected in parallel for AC. The resistance of resistor R_e is enhanced by $1+\beta$ at base (see emitter follower input resistance). The total input resistance of the common emitter amplifier is $R_1 \,||\, R_2 \,||\, R_e(1+\beta)$. Capacitor C_1 and the input resistance form a high pass filter with a cut-off angular frequency $\omega = 1/([R_1 \,||\, R_2 \,||\, R_e]C)$. The output resistance of the common emitter amplifier is approximately equal to R_c.

The selection criteria for R_1 and R_2 are the same as for the emitter follower.

1. $R_1 \,||\, R_2$ should be 10 times smaller than $R_e(1+\beta)$. In this case the voltage divider is sufficiently independent on the presence of the transistor.
2. The ratio R_1 / R_2 is chosen to set the DC currents.

The choice of the operating collector voltage is $V_c = 0.5V_{CC}$ (this determines the collector current and the emitter current if the base current contributing to the emitter current is neglected).

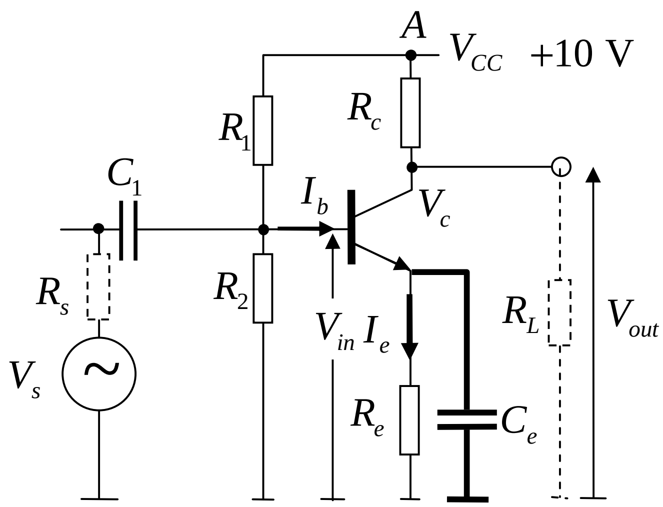


Fig. 46. Low AC signal common emitter amplifier.

Addition of the capacitor C_e increases the gain of the common emitter amplifier to a maximum (for AC signals) but generally speaking this gain is highly nonlinear because the intrinsic resistance of the transistor depends on the collector current.

The AC gain is

$$A \approx \frac{R_c}{r_e}$$

However, if the amplitude of the input voltage is very small this nonlinearity can be neglected (because the collector current will change very little).

The choice of C_e affects $f_{\min}$ and is similar to the choice of C_1 (see the emitter follower analysis).

Miller effect.

Miller effect is a confusing way of a circuit analysis where the voltages change at both ends of a capacitor synchronously. In a certain sense this may increase the apparent value of the capacitor. Compare two circuits in Fig. **47**

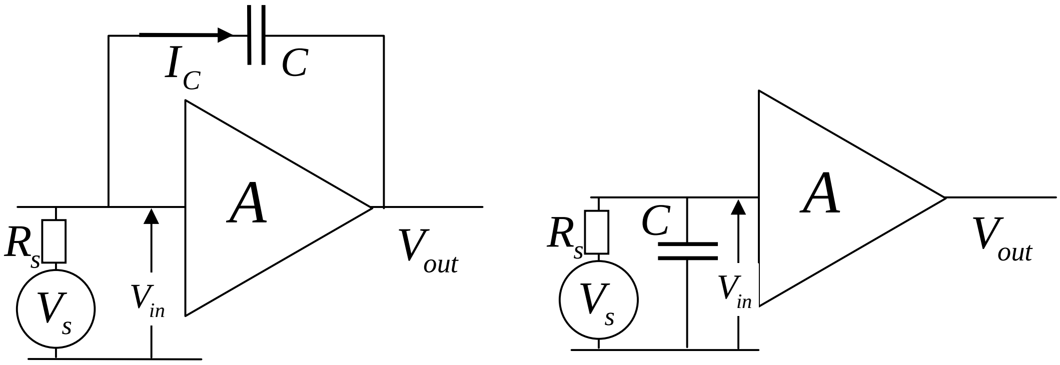


Fig. 47. Miller effect.

For the first circuit $V_{out} = AV_{in}$ and $I_C = (V_{in} - V_{out})\, j\omega C = V_{in}(1-A)\, j\omega C$. For the second circuit $I_C = V_{in}\, j\omega C$. In the first circuit, the voltage source "sees" a $1-A$ times larger effective capacitor $C_{eff} = (1-A)C$. The input voltage is $V_{in} = V_s \dfrac{1}{j\omega R_s C_{eff} + 1}$ and is much smaller than V_s if $\omega R_s C_{eff} \gg 1$. This effect plays an important role in common emitter amplifiers because it limits the maximum frequency which can be amplified.

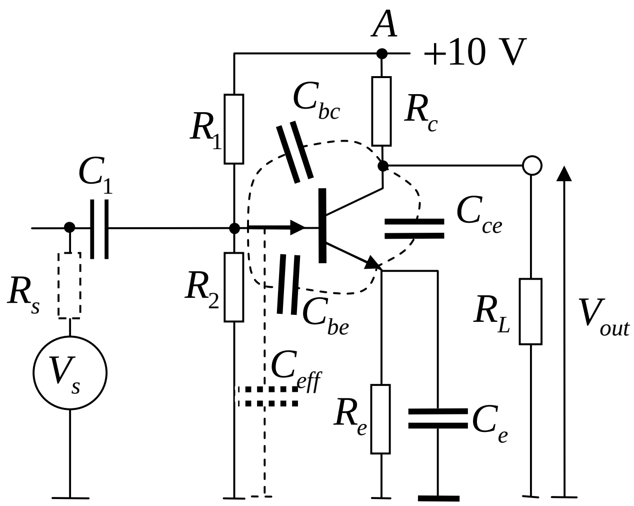


Fig. 48. Low AC signal common emitter amplifier at high frequency**.**

The base-collector capacitor is on the order of 5 pF but the effect of this capacitor is much larger because of the Miller effect. The effective capacitance of C_{bc} is increased proportionally to the gain of the common emitter amplifier. If the amplification of the common emitter amplifier is –200, the effective capacitance C_{eff} is 1000 pF. The grounded effective capacitor is connected parallel to R_2. If the internal resistance of the signal source is 50 Ω, the frequency at which the absolute value of the impedance of C_{eff} equals the resistance of R_s can be determined from the equation $1/2\pi f C_{eff} = R_s$. Solving for f gives the result

$$f = \frac{1}{2\pi R C_{eff}} = \frac{1}{2\pi R\left(1-A\right)C_{be}} = \frac{1}{2\pi \cdot 50\Omega \cdot \left(1+200\right)\cdot 5\text{ pF}} \approx 3\text{ MHz}.$$

Note that the bandwidth of this amplifier is inversely proportional to its amplification. This is a quite general principle. Recollect the analysis of the inverting amplifier based on an operational amplifier. The bandwidth of the closed loop amplification is also inversely proportional to the value of the closed loop amplification. There is always a trade-off between the bandwidth and the gain value.

To increase the bandwidth, a common base amplifier can be used.

Common base amplifier

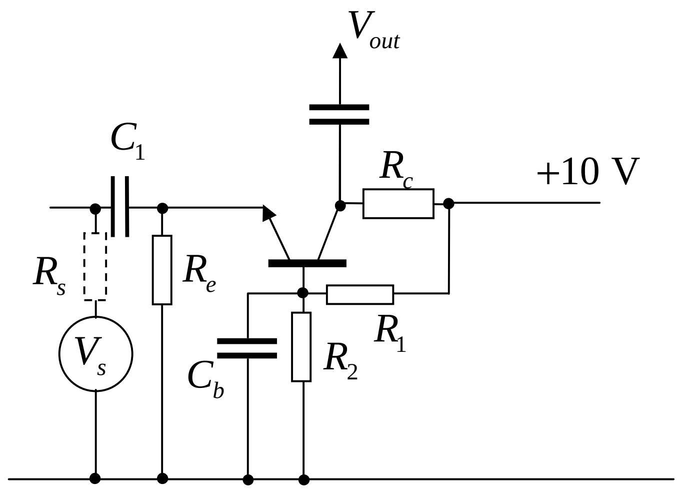


Fig. 49. Common base amplifier.

In this amplifier the base is grounded (see capacitor C_b) and therefore there is no increase in the effective value of the collector-base capacitor (no Miller effect). A drawback of this circuit is a very small input impedance (note that the input current is the emitter current not the base current as in the case of common emitter amplifier. Therefore a common base amplifier is frequently used together with an emitter follower (can you explain why?) as shown in Fig. **50**.

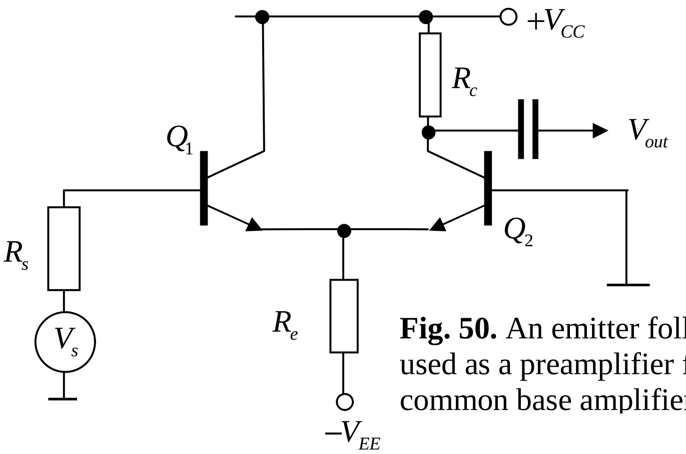


Fig. 50. An emitter follower is used as a preamplifier for a common base amplifier.

Lectures 10-11

Transmission lines

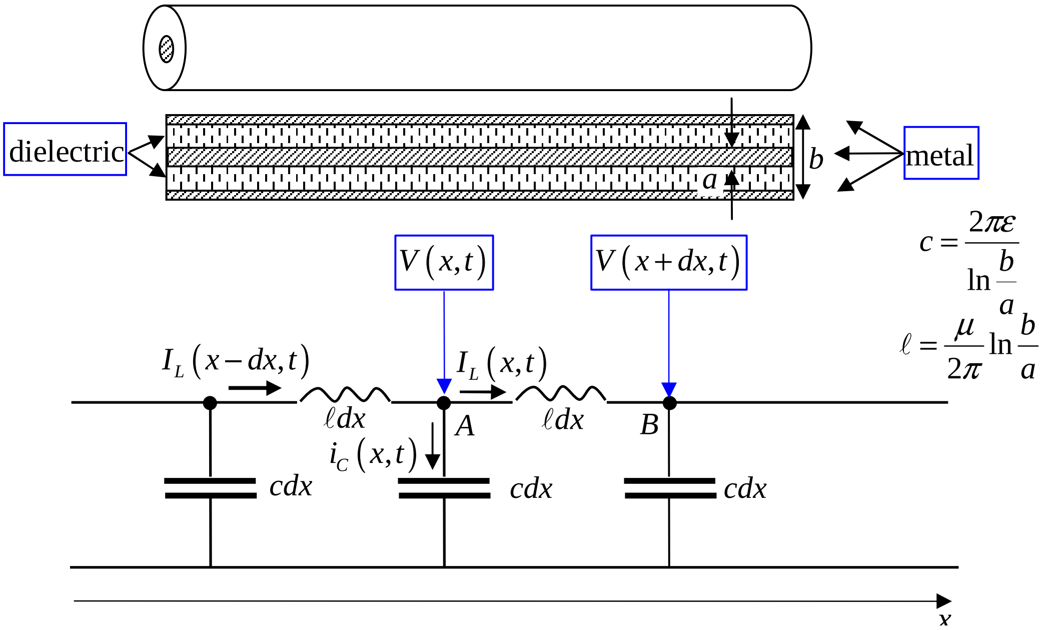


Fig. 51. Transmission line and its equivalent circuit**.**

Our goal is to derive an equation describing the dependence of the voltages and currents on time *t* at different locations *x*. This will be a wave equation. Wave equations are very important. They appear in very different fields of physics (acoustics, quantum mechanic, electrodynamics). The physical model of the transmission line consists of inductors (connected in series) and capacitors (see Figure). The two quantities ℓ and c are the inductance and capacitance per unit length of the transmission line.

Derivation of the wave equation

We start from the Kirchhoff current law at node *A* which reads

$$I_L(x-dx,t) = i_c(x,t) + I_L(x,t) \tag{8}$$

Solving this for i_c, one gets

$$i_c(x,t) = -\frac{\partial}{\partial x} I_L(x,t)\,dx \tag{9}$$

Voltage across the capacitor is related to the charge on the capacitor $q_c(x,t)$ and the capacitance cdx according to

$$V(x,t)=\frac{q_c(x,t)}{cdx} \tag{10}$$

We take partial time derivatives of both sides

$$\frac{\partial}{\partial t}V(x,t)=\frac{\partial}{\partial t}\frac{q_c(x,t)}{cdx}=\frac{i_c(x,t)}{cdx} \tag{11}$$

and substitute this into Eq. (9)

$$\frac{\partial}{\partial t}V(x,t)=-\frac{1}{c}\frac{\partial}{\partial x}I_L(x,t) \tag{12}$$

Voltage V_{AB} across the inductance equals

$$V(x+dx,t)-V(x,t)=-\ell dx\frac{\partial}{\partial t}I_L(x,t) \tag{13}$$

Replacing the left hand side with a partial derivative

$$\frac{\partial}{\partial x}V(x,t)dx=-\ell dx\frac{\partial}{\partial t}I_L(x,t) \tag{14}$$

and dividing both sides by dx one gets

$$\frac{\partial}{\partial x}V(x,t)=-\ell\frac{\partial}{\partial t}I_L(x,t) \tag{15}$$

We differentiate Eq. (12) and Eq. (15) over time and coordinate respectively

$$\frac{\partial^2}{\partial x\partial t}I_L(x,t)=-\frac{1}{\ell}\frac{\partial^2}{\partial x^2}V(x,t) \tag{16}$$

$$\frac{\partial^2}{\partial t\partial x}I_L(x,t)=-c\frac{\partial^2}{\partial t^2}V(x,t) \tag{17}$$

The left hand sides of these two equations are equal. Equality of the right hand sides leads to the following wave equation for the voltage.

$$\frac{\partial^2}{\partial x^2}V(x,t)=\ell c\frac{\partial^2}{\partial t^2}V(x,t) \tag{18}$$

A similar equation can be obtained for currents.

It easy to verify that any function $V\left(t\pm[\ell c]^{1/2}x\right)$ satisfies the wave equation.

First we calculate the right hand side of Eq. (18)

$$\frac{\partial}{\partial t}V\left(t\pm[\ell c]^{1/2}x\right)=V'\left(t\pm[\ell c]^{1/2}x\right)\frac{\partial}{\partial t}\left(t\pm[\ell c]^{1/2}x\right)=V'\left(t\pm[\ell c]^{1/2}x\right), \tag{19}$$

where $V'(u)$ stands for the derivative of the function $V(u)$. The second partial derivative can be obtained by differentiation of the above equation over time

$$\frac{\partial^2}{\partial t^2}V\left(t\pm[\ell c]^{1/2}x\right)=V''\left(t\pm[\ell c]^{1/2}x\right)=V''\left(t\pm[\ell c]^{1/2}x\right) \tag{20}$$

Now we calculate the left hand side of Eq. (18)

$$\frac{\partial}{\partial x}V\left(t\pm[\ell c]^{1/2}x\right)=V'\left(t\pm[\ell c]^{1/2}x\right)\frac{\partial}{\partial x}\left(t\pm[\ell c]^{1/2}x\right)=\pm[\ell c]^{1/2}\cdot V'\left(t\pm[\ell c]^{1/2}x\right) \tag{21}$$

$$\frac{\partial^2}{\partial x^2}V\left(t\pm[\ell c]^{1/2}x\right)=\ell c\cdot V''\left(t\pm[\ell c]^{1/2}x\right) \tag{22}$$

Comparison of the left hand side to the right hand side shows that the wave equation is satisfied.

Relation between current and voltage in the transmission line

Form equation (12) follows the following equality

$$I_L(x,t)=-c\int\frac{\partial}{\partial t}V(x,t)dx \tag{23}$$

If we substitute $V(x,t)=V\left(x\pm(\ell c)^{-1/2}t\right)$ we will get

$$I_L\left(t\pm(\ell c)^{1/2}x\right)=-c\int V'\left(t\pm(\ell c)^{1/2}x\right)dx=\mp\frac{c}{\sqrt{\ell c}}\int\frac{d}{du}V'(u)du=\mp\left(\frac{c}{\ell}\right)^{1/2}V\left(t\pm(\ell c)^{1/2}x\right).$$

Therefore $\dfrac{V}{I_L}=\mp\left(\dfrac{\ell}{c}\right)^{1/2}$. The value $\left(\dfrac{\ell}{c}\right)^{1/2}$ has the dimension of a resistance (Ω) and is called an impedance of a transmission line. Standard impedance for transmission lines is 50 Ω. Note that the plus corresponds to the wave propagating to the left (see next section).

Pulse propagation in a transmission line.

If we draw function $V(t \pm x/\upsilon)$, where $\upsilon = 1/\sqrt{\ell c}$ at different but fixed times, the function repeats itself shifted to the left (if the sign is "minus") or to the right (the sign is "plus").

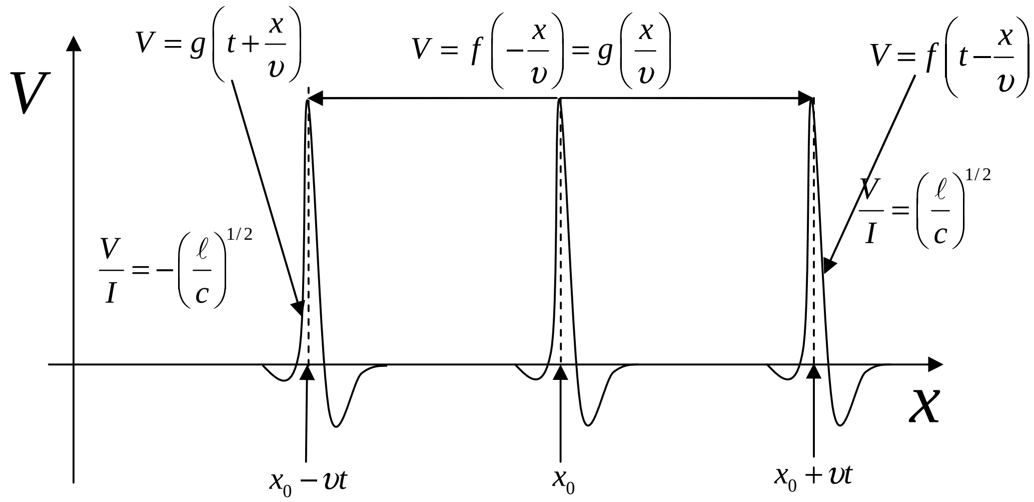


Fig. 52. Three snapshots of a pulse at zero time and at time *t*. The pulse propagates along *x* axis with speed υ. The ratio V/I depends on the direction of propagation.

Note that $f\left(t - \frac{x_0 + \upsilon t}{\upsilon}\right) = f\left(-\frac{x_0}{\upsilon}\right)$

$$g\left(t + \frac{x_0 - \upsilon t}{\upsilon}\right) = g\left(\frac{x_0}{\upsilon}\right)$$

Reflection from a transmission line loaded by an impedance Z.

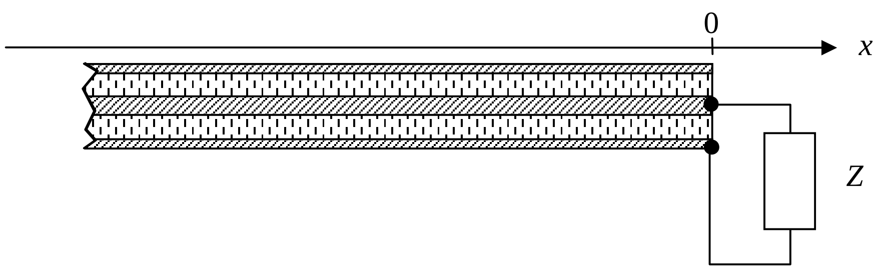


Fig. 53. Loaded transmission line.

At the point where the load is connected to the transmission line the ratio of the voltage to the current is equal to the value of the impedance.

Consider two **cos**-waves propagation in opposite directions.

$$V_{\rightarrow} = V_0 \exp\left(j\omega t - j\frac{\omega}{\upsilon}x \right) \equiv V_0 \exp\left(j\omega t - jkx \right),\ I_{\rightarrow} = \frac{V_0}{Z_{TL}} \exp\left(j\omega t - jkx \right)$$

and

$$V_{\leftarrow} = V_1 \exp\left(j\omega t + jkx \right),\ I_{\leftarrow} = -\frac{V_1}{Z_{TL}} \exp\left(j\omega t + jkx \right)$$

where $k \equiv \omega / \upsilon$ is called a wave number. The ratio of the total voltage to the total current is given by

$$\frac{V}{I} = Z_{TL} \frac{V_0 \exp\left(j\omega t - jkx \right) + V_1 \exp\left(j\omega t + jkx \right)}{V_0 \exp\left(j\omega t - jkx \right) - V_1 \exp\left(j\omega t + jkx \right)} = Z_{TL} \frac{V_0 \exp\left(-jkx \right) + V_1 \exp\left(jkx \right)}{V_0 \exp\left(-jkx \right) - V_1 \exp\left(jkx \right)}$$

When $x = 0$ this ratio becomes

$$\frac{V}{I} = Z_{TL} \frac{V_0 + V_1}{V_0 - V_1}$$

We can choose the values of V_0 and V_1 to satisfy the condition $V / I = Z_L$

$$Z_L = Z_{TL} \frac{V_0 + V_1}{V_0 - V_1}$$

$$Z_L V_0 - Z_L V_1 = Z_{TL} V_0 + Z_{TL} V_1$$

$$V_1 = \frac{Z_L - Z_{TL}}{Z_L + Z_{TL}} V_0$$

Therefore to satisfy the condition $V / I = Z_L$ at the point $x = 0$ the counter propagating wave should have a complex amplitude

$$V_1 = \frac{Z_L - Z_{TL}}{Z_L + Z_{TL}} V_0$$

The solution $V_0 \exp\left(j\omega t - jkx \right) + V_1 \exp\left(j\omega t + jkx \right)$ satisfies the wave equation everywhere and the boundary condition at the point where the load is connected. Therefore $V_0 \exp\left(j\omega t - jkx \right) + V_1 \exp\left(j\omega t + jkx \right)$ is the solution correctly describing the voltage at any point of the semi-infinite transmission line (including its end with the load) shown in Fig. 45.

Examples.

1. $Z_L = Z_{TL}$

In this case the amplitude of the counter propagating wave is zero. A pulse sent through a line with a load.

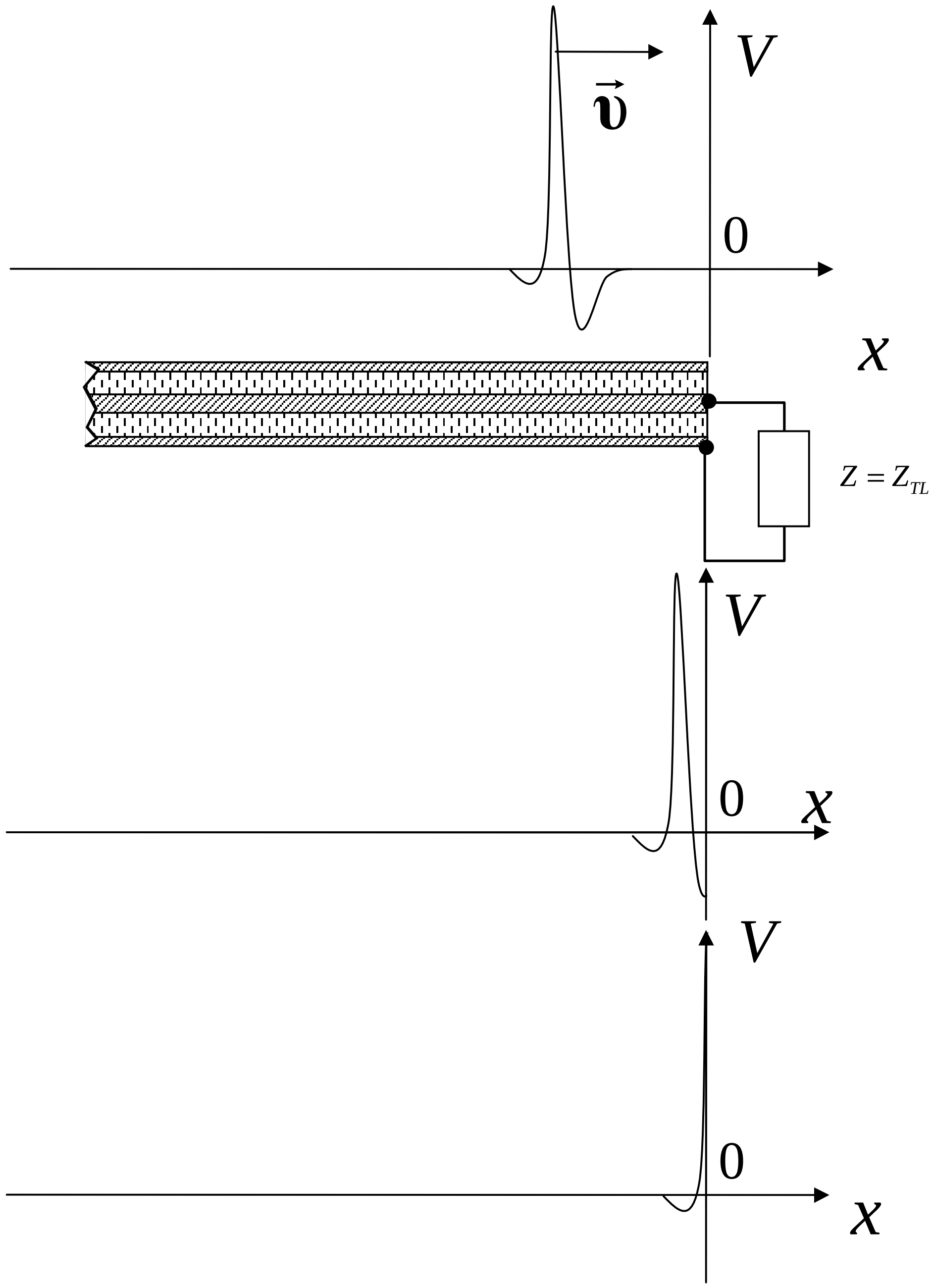


Fig. 54. No reflection from the load.

2. $Z_L = 0$

In this case the reflection coefficient is frequency independent and the amplitude of the counter propagating wave equals the amplitude of the wave propagating to the right. A pulse sent through a transmission live will be reflected by the shortcut on the line end and will propagate to the left.

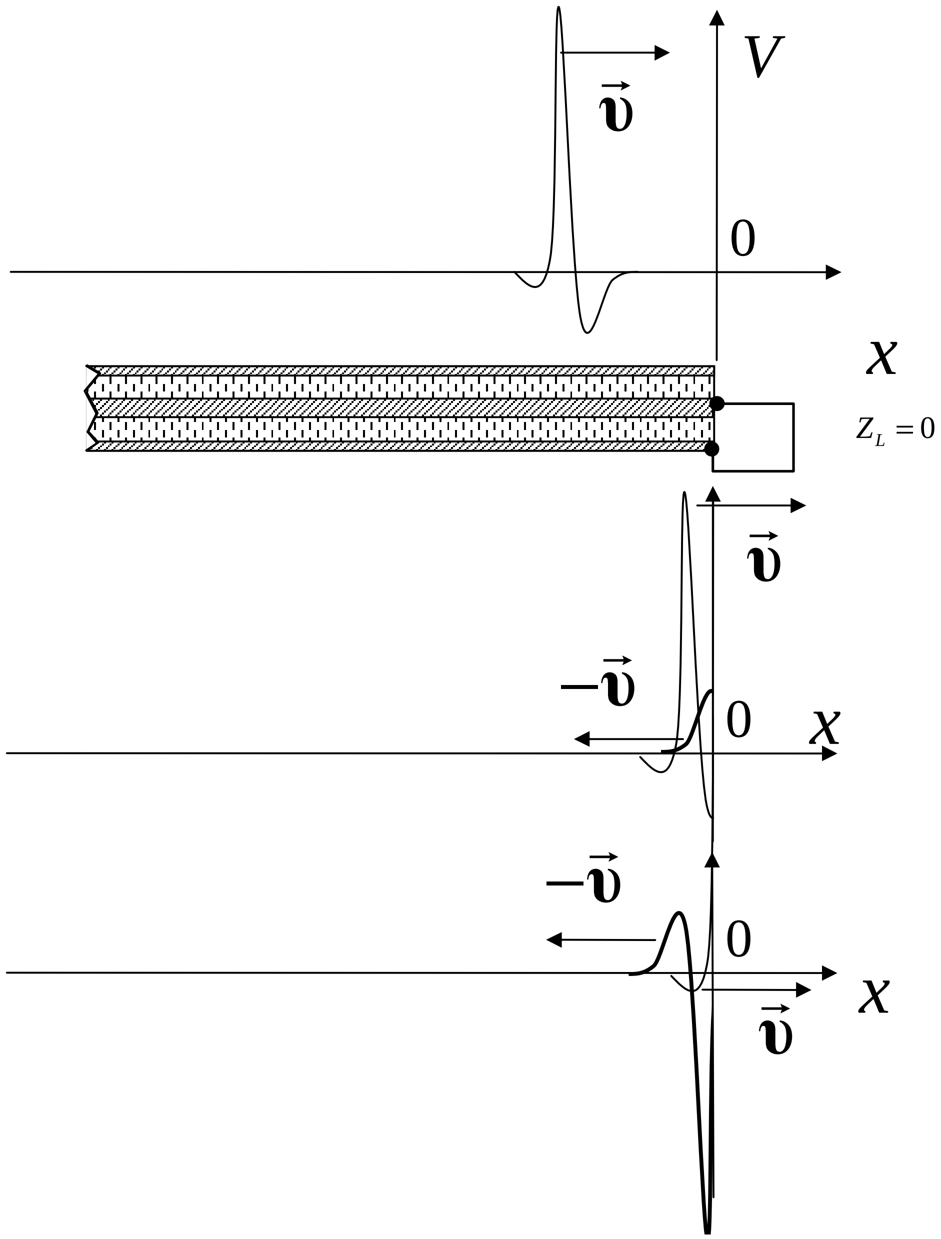


Fig. 55. Reflection from a shortcut.

3. $Z_L = \infty$

In this case the amplitude of the counter propagating wave has the same absolute value but opposite sign. A reflected pulse will have an opposite amplitude to the amplitude of the incoming pulse.

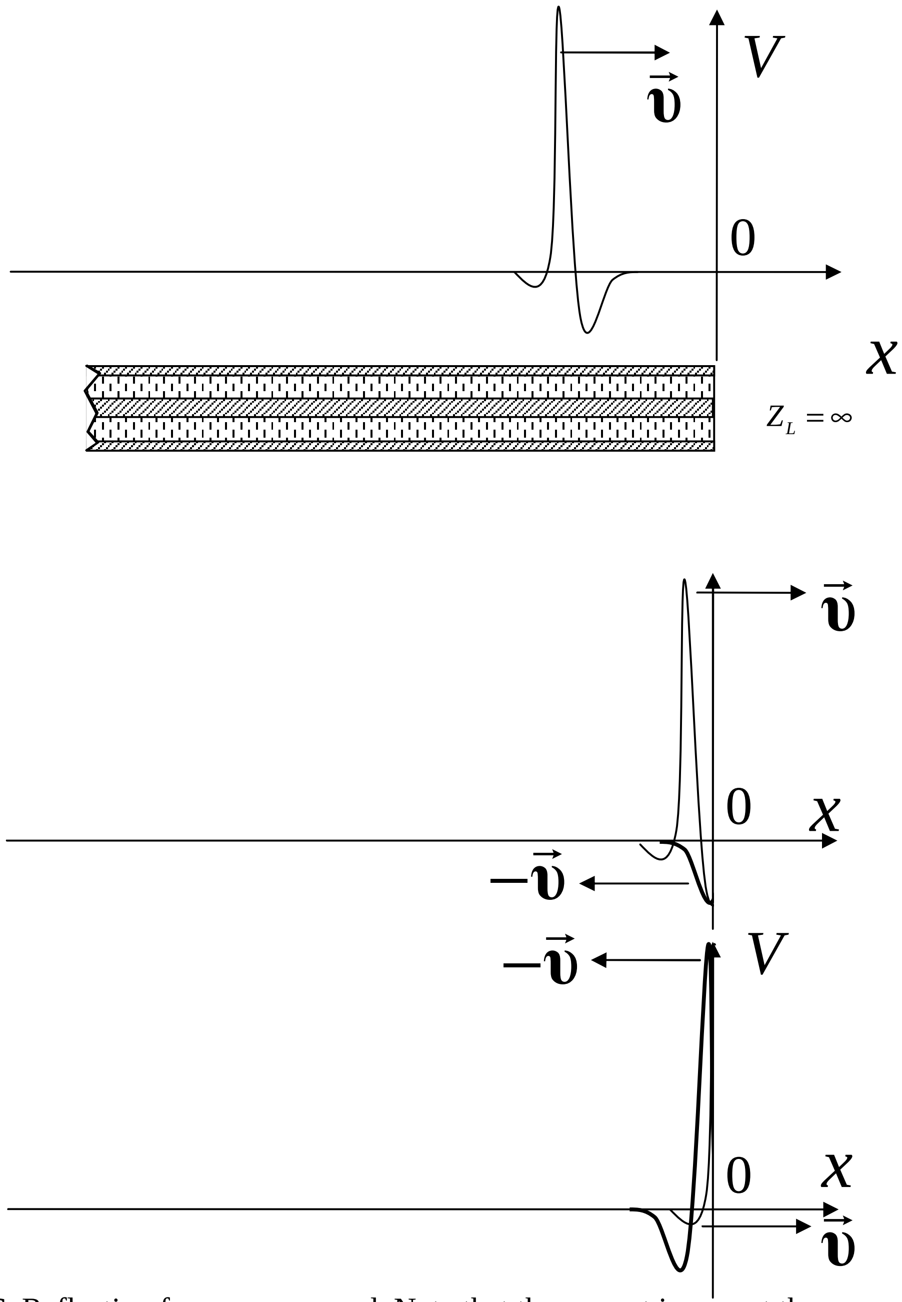


Fig. 56. Reflection from an open end. Note that the current is zero at the open end.

Remember! The total voltage is the sum of the two pulses.

4. How large is the reflection from a load of 100 Ω connected to a transmission line with impedance of 50 Ω.

$$V_{ref} = \frac{Z_L - Z_{TL}}{Z_L + Z_{TL}} V_{in} = \frac{100-50}{100+50} V_{in} = \frac{1}{3} V_{in}$$

Input impedance of a transmission line with a load

$$Z \equiv \frac{V}{I} = Z_{TL} \frac{V_0 \exp(-jkx) + V_1 \exp(jkx)}{V_0 \exp(-jkx) - V_1 \exp(jkx)}$$ and we substitute $V_1 = \frac{Z_L - Z_{TL}}{Z_L + Z_{TL}} V_0$

$$Z = Z_{TL} \frac{V_0 \exp(-jkx) + \frac{Z_L - Z_{TL}}{Z_L + Z_{TL}} V_0 \exp(jkx)}{V_0 \exp(-jkx) - \frac{Z_L - Z_{TL}}{Z_L + Z_{TL}} V_0 \exp(jkx)} = Z_{TL} \frac{(Z_L + Z_{TL})\exp(-jkx) + (Z_L - Z_{TL})\exp(jkx)}{(Z_L + Z_{TL})\exp(-jkx) - (Z_L - Z_{TL})\exp(jkx)} =$$

$$= Z_{TL} \frac{Z_L \cos kx - jZ_{TL} \sin kx}{Z_{TL} \cos kx - jZ_L \sin kx} = Z_{TL} \frac{Z_L \cos kl + jZ_{TL} \sin kl}{Z_{TL} \cos kl + jZ_L \sin kl}$$

where l is the length of the line (a positive number). Note that the x coordinate is $-l$ at the beginning of the line.

Examples.

1. Input impedance of an open end line

$$Z_{OE} = Z_{TL} \frac{Z_L \cos kl + jZ_{TL} \sin kl}{Z_{TL} \cos kl + jZ_L \sin kl} = Z_{TL} \frac{\cos kl}{j \sin kl}$$

2. Input impedance of a line with a shortcut at the end

$$Z_{SC} = Z_{TL} \frac{Z_L \cos kl + jZ_{TL} \sin kl}{Z_{TL} \cos kl + jZ_L \sin kl} = jZ_{TL} \frac{\sin kl}{\cos kl}$$

Interesting that $Z_{OE} Z_{SC} = Z_{TL}^2$ independent of the length

You can use this relation to measure Z_{TL}, the impedance of an infinitely long transmission line.

Lectures 12-14.

Formal logic. Formal logic identities.

Binary Boolean operations are defined by

$\overline{1}=0$, $\overline{0}=1$, $A\cdot 1=A$, $A\cdot 0=0$, $A+1=1$, $A+0=A$,

$A\cdot A=A$, $A+A=A$, $\overline{\overline{A}}=A$

Commutation axioms: $A+B=B+A$

$$A\cdot B=B\cdot A$$

If more than two operands are involved Boolean Algebra satisfies the following association and distribution laws

Association : $A+(B+C)=(A+B)+C\equiv A+B+C$

$$A\cdot(B\cdot C)=(A\cdot B)\cdot C\equiv A\cdot B\cdot C$$

Distribution: $A\cdot(B+C)=A\cdot B+A\cdot C$

$$A+(B\cdot C)=(A+B)\cdot(A+C)$$

Exercise 1: Show that $A+(A\cdot B)=A$ and $A\cdot(A+B)=A$

Solution: $A+(A\cdot B)=A\cdot(1+B)=A\cdot 1=A\quad A+A=A$

$$A\cdot(A+B)=A\cdot A+A\cdot B=A+AB=A(1+B)=A\cdot 1=A$$

DeMorgan's theorems: $\overline{A}\cdot\overline{B}=\overline{A+B}$ **and** $\overline{A\cdot B}=\overline{A}+\overline{B}$

Logic identities can be proved by using a truth table.

Truth table (examples)

A	B	$A\cdot B$	$\overline{A\cdot B}$	$\overline{A}$	$\overline{B}$	$\overline{A}\cdot\overline{B}$	$\overline{A}+\overline{B}$	$A+B$	$\overline{A+B}$
0	0	0	**1**	1	1	**1**	**1**	0	**1**
0	1	0	**1**	1	0	**0**	**1**	1	**0**
1	0	0	**1**	0	1	**0**	**1**	1	**0**
1	1	1	**0**	0	0	**0**	**0**	1	**0**

We have derived an important de Morgan's theorem

$\overline{A}\cdot\overline{B}=\overline{A+B}$. Consequence: An OR operation can be replaced by a combination of AND and NOT operations. This is given by the equality $A+B=\overline{\overline{A}\cdot\overline{B}}$

In a similar way one can prove that $\overline{A\cdot B}=\overline{A}+\overline{B}$ and therefore $A\cdot B=\overline{\overline{A}+\overline{B}}$

Exercise 2. Show that $(A+B)\cdot(A+C)=A+BC$

Solution. $(A+B)\cdot(A+C)=(A+B)\cdot A+(A+B)\cdot C=AA+BA+AC+BC=$

$$=A(1+B)+AC+BC=A1+AC+BC=A+AC+BC=A(1+C)+BC=$$
$$=A+BC$$

Any numerical computation can be represented by a very large truth table. Any truth table can be reduced to a long Boolean logic expression involving, for example, only AND and NOT operations. This also can be done using OR and NOT operations.

Digital electronics.

AND, OR, NOT gates and their truth tables.

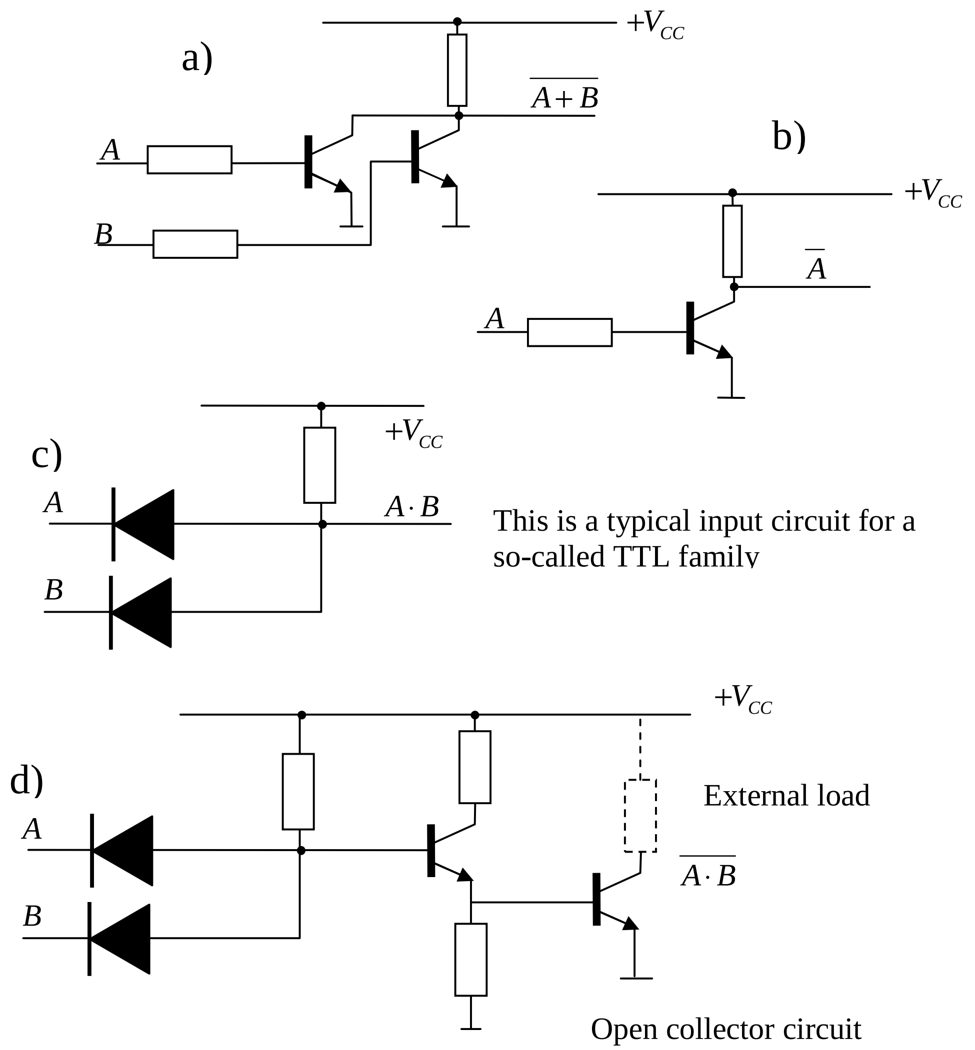


Fig. 57. Examples of logical circuits.

Convention:

Low voltage level is “false” or “zero”

High voltage level is “true” or “one”

AND gate. $OUTPUT = A \cdot B$ **(binary multiplication)**

A	B	OUTPUT
0	0	0
1	0	0

0	**1**	**0**
1	**1**	**1**

OR gate. $OUTPUT = A + B$**.** In the case of two operands, this is binary addition where the output shows the higher (most left) bit position.

A	**B**	**OUTPUT**
0	**0**	**0**
1	**0**	**1**
0	**1**	**1**
1	**1**	**1**

Exclusive OR gate XOR. $OUTPUT = A \oplus B$**.** In the case of two operands, this is binary addition where the output shows the lower (most right) bit position.

A	**B**	**OUTPUT**
0	**0**	**0**
1	**0**	**1**
0	**1**	**1**
1	**1**	**0**

Exercise 4. Verify that $A \oplus B = \overline{A}B + A\overline{B}$. Use the Boolean logic laws and show that $A \oplus B = (A + B)\left(\overline{AB}\right)$.

Solution. $(A+B)\left(\overline{AB}\right) = A\left(\overline{AB}\right) + B\left(\overline{AB}\right) = A\left(\overline{A} + \overline{B}\right) + B\left(\overline{A} + \overline{B}\right) =$

$0 + A\overline{B} + B\overline{A} + 0 = A\overline{B} + B\overline{A}$

NOT gate

A	**OUTPUT**
0	**1**
1	**0**

Symbolic representations of logical gates.

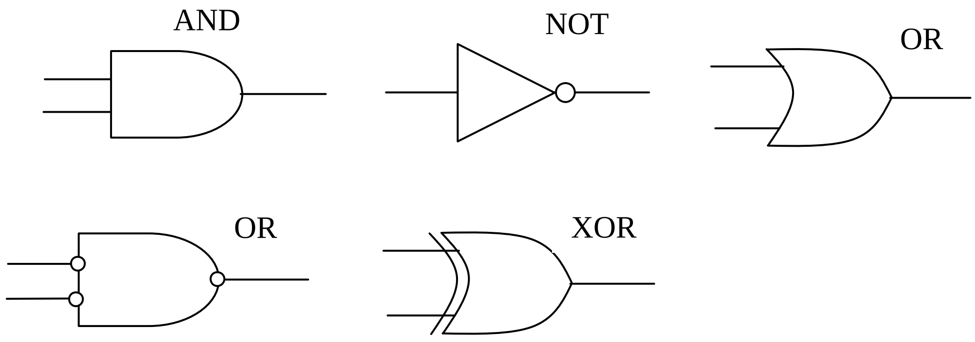


Fig. 58. Graphic representations of logical gates in electronic circuits**.**

TTL (transistor-transistor logic) and CMOS (complementary metal oxide semiconductor) ECL (emitter coupled logic) logic families. Three state logic.

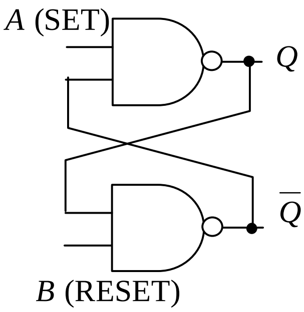
Basic Properties of some TTL Families.

	74 subfamily	74LS subfamily
Supply Voltage	+5V (+/- 0.5V)	+5V (+/- 0.5V)
‘1’ Level Output Current	0.4mA	0.4mA
‘0’ Level Output Current	16mA	8mA
‘1’ Level Input Voltage (min)	2V	2V
‘1’ Level Input Voltage (typical)	3.5V	3.5V
‘0’ Level Input Voltage (max)	0.8V	0.8V
‘0’ Level Input Voltage (typical)	0.35V	0.35V
‘1’ Level Input Current	0.04mA	0.05mA
‘0’ Level Input Current	1.6mA	0.4mA
Time delay	10 ns	2 ns

Note. An input which is not connected to anything works like HIGH for the TTL family.

Three state logic has a state when the output is disconnected (disabled). This type of circuits is used, for example, when many devices use shared data bus for communications.

SR Flip-flops.

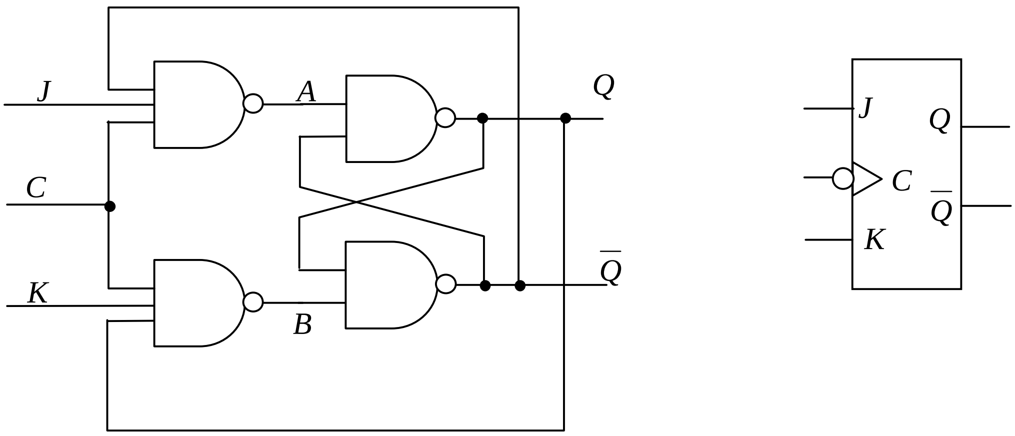


A	*B*	*Q*	$\overline{Q}$
1	**0**	**0**	**1**
0	**1**	**1**	**0**
0	**0**	**1**	**1**
1	**1**	**1** **0**	**0** **1**

Fig. 59. Flip-flop circuit and its truth table

SR (Set-Reset) flip-flop has two inputs (SET and RESET). Note that when the values of A and B are 1, there two possible values for the output terminals. If B is kept at high, sending zero to A sets the value Q at high (one). This state will be maintained until both inputs are set to low (zero). Similarly, if A is at high, sending zero to B will make the value of Q low (zero). This state also will be kept by the circuit until both inputs are set to zero.

This flip-flop can be used to build a more popular JK flip-flop shown in the next Figure.



J	K	Q_n	$\overline{Q}_n$	C (clock)	A	B	Q_{n+1}	$\overline{Q}_{n+1}$	
1	**1**	**1**	**0**	$1 \to 0$	**1**	$1 \to 0$	**0**	**1**	Q and $\overline{Q}$ flip on every clock pulse
1	**1**	**0**	**1**	$1 \to 0$	$1 \to 0$	**1**	**1**	**0**	
0	**0**	**1**	**0**	$1 \to 0$	**1**	**1**	**1**	**0**	No change for Q and $\overline{Q}$
0	**0**	**0**	**1**	$1 \to 0$	**1**	**1**	**0**	**1**	
1	**0**			$1 \to 0$		**1**	**1**	**0**	J and K are moved to Q and $\overline{Q}$ respectively
0	**1**			$1 \to 0$	**1**		**0**	**1**	

Fig. 60. JK flip-flop and its truth table.

Counters.

JK flip-flops can be used to build a binary counter.

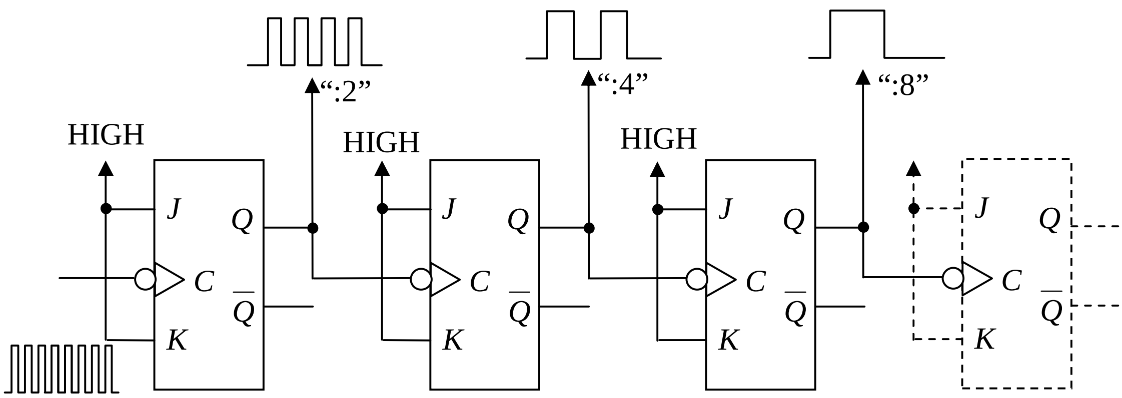


Fig. 61. 4-bit counter.

Pulse generator.

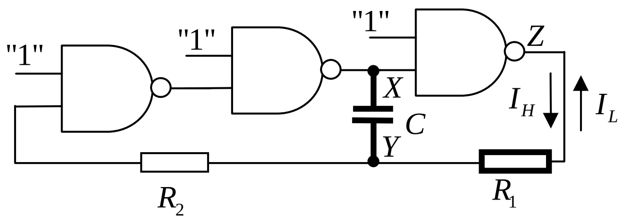


Fig. **62**. Pulse generator.

Three invertors connected in series invert the input signal and if the output of the last inverter is connected to the input of the first inverter a logical contradiction appears. The circuit starts to oscillate with a frequency of about 20 MHz "trying to resolve" the contradiction. The capacitor and the resistor (see Fig.) partially resolve the contradiction. When V_X is LOW and V_Z is HIGH, potential at *Y* remains lower than V_Z by the potential drop across resistor R_1 (see the current direction shown in the Fig.). No contradiction appears as long as V_Y stays at logical LOW. However, since capacitor C is charging, potential V_Y increases.

When $V_Y = V_{HT}$ and the potential is high enough to be logical HIGH, 1) V_X will be set to HIGH due to the first two invertors, 2) V_Y will be pulled to even a higher potential $V_Y \approx V_{HT} + V_H$, and 3) V_Z jumps to LOW , 4) the current through R_1 is reversed.

V_Y starts to decrees until $V_Y = V_{LT}$ and then 1) V_X jumps to LOW, 2) V_Y jumps to $V_Y \approx V_{LT} - V_H$ 3) V_Z jumps to HIGH, 4) the current through R_1 is reversed and the circle stars over. The period of oscillations is given by $T \approx 2.2 R_1 C$

DAC and ADC

Digital-Analogue Converter (DAC) and Analogue-Digital Converter are electronic circuits which provide interconnection between analogue and digital circuits.

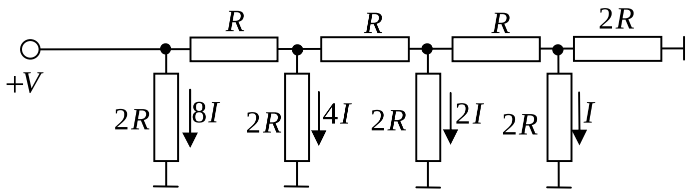


Fig. 64. 2*R*-*R* circuit. Prove that currents through the 2*R* resistors are two times larger in each stage from right to left. Hint: Use Kirchhoff current law starting from the most right pair of the 2*R* resistors and then evaluate the equivalent resistance of the three most right resistors.

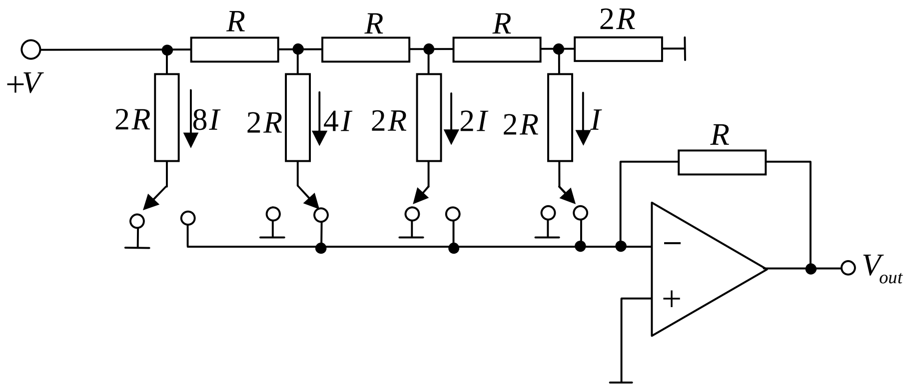


Fig. 64. Show that $V_{out} = -(1+4)V/16$ if the switches are in the position as shown. Show that $V_{out} = -(1+2+4+8)V/16$ if all switches are in the "right" position.

The circuit in Fig. **64** works as a DAC (Digital-to-Analogue Converter).

ADCs are complementary to DACs. DACs and ADCs are used as bridging circuits connecting analogue outputs (for example, analogue outputs of sensor signals) to digital electronics and digital outputs to analogue inputs. A simple ADC can be made of a comparator (a circuit which signals when two voltages are equal), a binary counter, and DAC. Try to figure out how.

Lecture 15-18

Integral transformations (Fourier and Laplace transformations). Techniques for solving differential equations.

Fourier transformation. Fourier transform of a function $f(t)$ is a function of a different variable ω calculated according to the following expression

$$F_\omega\left[f(t)\right] \equiv \int_{-\infty}^{\infty} f(t)\cdot\exp(-j\omega t)dt$$

provided that the integral exists (converges).

Laplace transformation. Fourier transform of a function $f(t)$ is a function of a different variable s calculated according to the following expression

$$L_s\left[f(t)\right] \equiv \int_{0}^{\infty} f(t)\cdot\exp(-st)dt$$

provided that the integral exists (converges).

Some properties of these transforms are very similar but some are different. The main difference between these transform is the restrictions on the properties of the function $f(t)$ imposed by the existence (convergence) of the corresponding integrals. Both transforms are very useful for solving differential equations. The usage of these transformations is based on the following theorems

Theorem 1. Transform of a shifted function.

$$F_\omega\left[f(t-t_0)\right] = \exp(j\omega t_0)F_\omega\left[f(t)\right]$$

$$L_s\left[f(t-t_0)\theta(t-t_0)\right] = \exp(-st_0)L\left[f(t)\theta(t)\right]$$

Proof .

3.1. $\int_{-\infty}^{\infty} f(t-t_0)\exp(-j\omega t)dt = \int_{-\infty}^{\infty} f(u)\exp(-j\omega[u+t_0])du$, where we have used substitution $t = u + t_0$.

Because $\int_{-\infty}^{\infty} f(u)\exp(-j\omega[u+t_0])du = \exp(j\omega t_0)\int_{-\infty}^{\infty} f(u)\exp(-j\omega u)du$

3.2. $\int_{0}^{\infty} f(t-t_0)\theta(t-t_0)\exp(-j\omega t)dt = \int_{0}^{\infty} f(u)\exp(-s[u+t_0])du$, where we have used substitution $t=u+t_0$.

Because $\int_{0}^{\infty} f(u)\exp(-s[u+t_0])du = \exp(-st_0)\int_{0}^{\infty} f(u)\exp(-su)du = \exp(-st_0)L_s[f(t)]$

Theorem 2. Derivative of a transform.

$$\frac{dF_\omega[f]}{d\omega} = -jF_\omega(t\cdot f)$$

$$\frac{dL_s[f]}{ds} = -L_s[t\cdot f]$$

Proof.

$$\frac{d}{d\omega}\int_{-\infty}^{\infty} f(t)\exp(-j\omega t)dt = \int_{-\infty}^{\infty} f(t)\frac{d}{d\omega}\exp(-j\omega t)dt = -j\int_{-\infty}^{\infty} tf(t)\exp(-j\omega t)dt$$

$$\frac{d}{ds}\int_{0}^{\infty} f(t)\exp(-st)dt = \int_{0}^{\infty} f(t)\frac{d}{ds}\exp(-st)dt = -\int_{0}^{\infty} tf(t)\exp(-st)dt$$

Theorem 3. Transform of a derivative

$$F_\omega[f'] = j\omega F_\omega[f]$$

$$F_\omega[f''] = (j\omega)^2 F_\omega[f]$$

$$L_s[f'] = sL_s[f] - f(0)$$

$$L_s[f''] = s^2 L_s[f] - sf(0) - f'(0)$$

Direct consequence of this theorem is transform of an integral

$$L_s\left[\int_0^t f(\tau)d\tau\right] = \frac{L_s[f]}{s}$$

Proof .

4.1.

$$\int_{-\infty}^{\infty}\frac{df(t)}{dt}\exp(-j\omega t)dt = f(t)\exp(-j\omega t)\Big|_{t=-\infty}^{t=\infty} - \int_{-\infty}^{\infty} f(t)\frac{d\exp(-j\omega t)}{dt}dt = j\omega\int_{-\infty}^{\infty} f(t)\exp(-j\omega t)dt$$

It is essential that for the Fourier transform to exist, that the value of $f(t)$ is zero at $\pm\infty$. Therefore $f(t)\exp(-j\omega t)\Big|_{t=-\infty}^{t=\infty} = 0$

4.2.

$$\int_{0}^{\infty}\frac{df(t)}{dt}\exp(-st)dt = f(t)\exp(-st)\Big|_{0}^{t=\infty} - \int_{0}^{\infty} f(t)\frac{d\exp(-st)}{dt}dt = -f(0) + s\int_{-\infty}^{\infty} f(t)\exp(-st)dt$$

It is essential that for the Laplace transform to exist, the value of $f(t)\exp(-st)$ should be zero at $+\infty$. Therefore $f(t)\exp(-st)\Big|_{0}^{t=\infty} = -f(0)\Big|_{0}^{t=\infty}$.

Thus we obtain $L_s[f'] = sL_s[f] - f(0)$**.** Using this result twice one gets

$L_s[f''] = -f'(0) + sL_s[f'] = -f'(0) - sf(0) + s^2L_s[f]$.

4.3. We introduce a function $g(t) \equiv \int_0^t f(\tau)d\tau$, which has the following properties:

$g(0) = 0$ and $g'(t) = f(t)$. Therefore $L_s\left[g(t)\right] = \dfrac{L_s[g']}{s} = \dfrac{L_s[f]}{s}$.

Theorem 4. **Partial fraction expansion.**

This theorem is very useful when calculating an inverse Laplace transform of a ratio of two polynoms of *s*. The theorem states that the following decomposition is always possible.

$$\frac{a_ns^n + a_{n-1}s^{n-1} + ...a_0}{b_ns^n + b_{n-1}s^{n-1} + ...b_0} = \frac{a_ns^n + a_{n-1}s^{n-1} + ...a_0}{(s-s_1)^{k_1}(s-s_2)^{k_2}...(s-s_p)^{k_p}} =$$
$$= \sum_{m=1}^{k_1}\frac{b_{1m}}{(s-s_1)^m} + \sum_{m=1}^{k_2}\frac{b_{2m}}{(s-s_2)^m} + ... + \sum_{m=1}^{k_n}\frac{b_{pm}}{(s-s_p)^m}, \quad n < \sum_{m=1}^{p}k_m$$

In these expressions $s_1, s_2, s_3...$ are the roots of the denominator. These roots can be multiple ($k_1, k_2, k_3,...$ is the multiplicity of the corresponding roots). This looks complicated but becomes clearer if an example is considered.

Example: Find the partial fraction expansion for the following function $\dfrac{s^2+s+1}{(s-1)^2(s-2)}$

We want to find *a*, *b*, and *c* such that

$$\frac{s^2+s+1}{(s-1)^2(s-2)}=\frac{a}{(s-1)^2}+\frac{b}{s-1}+\frac{c}{s-2}.$$

First we do simple algebra

$$\frac{a(s-2)+b(s-1)(s-2)+c(s-1)^2}{(s-1)^2(s-2)}=\frac{s^2+s+1}{(s-1)^2(s-2)}$$

$$\frac{as-2a+bs^2-3bs+2b+cs^2-2cs+c}{(s-1)^2(s-2)}=\frac{s^2+s+1}{(s-1)^2(s-2)}$$

The equality holds for all *s* if the factors in front of corresponding powers of *s* are equal. For example, the factors multiplying s^2 should be equal. We have to solve three linear equations for *a*, *b*, and *c*.

$$\begin{cases} b+c=1 \\ a-3b-2c=1 \\ -2a+2b+c=1 \end{cases} \Rightarrow \begin{cases} b+c=1 \\ a-b=3 \\ -2a+b=0 \end{cases} \Rightarrow \begin{cases} b+c=1 \\ a-b=3 \\ -a=3 \end{cases} \Rightarrow \begin{cases} c=7 \\ b=-6 \\ a=-3 \end{cases}$$

$$\frac{s^2+s+1}{(s-1)^2(s-2)}=-\frac{3}{(s-1)^2}-\frac{6}{s-1}+\frac{7}{s-2}$$

Other useful relations

1. Substitution

$$L_s\left[\exp(at)\cdot f(t)\right]=\int_0^\infty \exp(at)\,f(t)\exp(-st)dt=\int_0^\infty f(t)\exp(-[s-a]t)du=L_{s-a}\left[f(t)\right]$$

if $s-a>0$.

2. Laplace transform and convolution

The following integral

$$\int_0^t f(t-u)\,g(u)\,du \equiv f*g$$

is called convolution. There is a relation between Laplace transform of a convolution and Laplace transforms of the functions *f* and *g*.

$$\int_0^\infty \left(\int_0^t f(t-u) g(u) du \right) \cdot e^{-st} dt = \int_0^\infty f(t) \cdot e^{-st} dt \cdot \int_0^\infty g(t) \cdot e^{-st} dt$$

$$L\left[\int_0^t f(t-u) g(u) du \right] \equiv L[g * f] = L\left[f(t) \right] \cdot L\left[g(t) \right]$$

$$\int_0^t f(t-u) g(u) du = L_t^{-1}\left[L_s\left[f(t) \right] \cdot L_s\left[g(t) \right] \right]$$

Examples.

1.

$$L_s\left[\sin(\omega t) \right] = L_s\left[\frac{\exp(j\omega t) - \exp(-j\omega t)}{2j} \right] = \frac{1}{2j}\left(\frac{1}{s - j\omega} - \frac{1}{s + j\omega} \right) = \frac{1}{2j}\left(\frac{s + j\omega - (s - j\omega)}{s^2 + \omega^2} \right) =$$

$$\frac{1}{2j} \frac{2j\omega}{s^2 + \omega^2} = \frac{\omega}{s^2 + \omega^2}$$

2.

$$L\left[(c + j \cdot b) \exp\left\{ (a + j \cdot b) t \right\} \right] = \int_0^\infty (c + j \cdot b) \exp\left\{ (a + j \cdot b) t \right\} \exp(-st) dt =$$

$$-\frac{c + j \cdot b}{a + j \cdot b - s} = \frac{c + j \cdot b}{s - (a + j \cdot b)}$$

Steps for solving differential equations using Laplace transform

1. Apply Laplace transform to both sides of the differential equation (use transform of derivatives theorem).
2. Solve the linear equation for the unknown Laplace transform.
3. Find the inverse Laplace transform. Use partial fraction expansion if required.

Example.

Solve differential equation $2f'' + 8f' + 10f = 0$, if $f'(0) = 0$ and $f(0) = 1$

1.

$$L_s\left[2f''+8f'+10f\right]=2s^2L_s\left[f\right]-2sf(0)-2f'(0)+8sL\left[f\right]-8f(0)+10L_s\left[f\right]=$$
$$2s^2L_s\left[f\right]-2s+8sL\left[f\right]-8+10L_s\left[f\right]=0$$

2.

$$L_s\left[f\right]=\frac{2s+8}{2s^2+8s+10}$$

3.

Find roots of $2s^2+8s+10$. $s_{1,2}=\dfrac{-4\pm\sqrt{4^2-2\cdot 10}}{2}=-2\pm j$.

$$L_s\left[f\right]=\frac{2s+8}{2(s+2-j)\left(s+2+j\right)}=\frac{s+4}{\left(s+2-j\right)\left(s+2+j\right)}=\frac{A}{s+2-j}+\frac{B}{s+2+j}$$

The equality $A\left(s+2+j\right)+B\left(s+2-j\right)=s+4$ should hold for any *s*. There are two ways to ensure this. One approach is to write the equality relations for corresponding coefficients on the right and left hand sides. Thus

$As+A\left(2+j\right)+Bs+B\left(2-j\right)=\left(A+B\right)s+A\left(2+j\right)+B\left(2-j\right)=s+4$ holds if

$$\begin{cases} A+B=1 \\ A\left(2+j\right)+B\left(2-j\right)=4 \end{cases}$$

It many cases an easier approach is to substitute $s=-2-j$ in $A\left(s+2+j\right)+B\left(s+2-j\right)=s+4$. This makes the coefficient before *A* zero and let us find *B*. Then we substitute $s=-2+j$ making the coefficient before *B* zero (in this way we will find *A*).

$$\begin{cases} B\left(-2-j+2-j\right)=-2-j+4 \\ A(-2+j+2+j)=-2+j+4 \end{cases}$$

$$\begin{cases} B=j+\dfrac{1}{2} \\ A=\dfrac{2+j}{2j}=-j+\dfrac{1}{2} \end{cases}$$

Thus we obtain

$$L_s\left[f\right]==\frac{\frac{1}{2}+j}{s+2-j}+\frac{\frac{1}{2}-j}{s+2+j}$$

Inverse Laplace transform (see table) gives

Answer: $f = e^{-2t}\left[\cos(t) + 2\sin(t)\right]$.

Laplace transform of the response of a circuit to a δ-impulse (the initial conditions are assume to be all zeros) is called a transfer function of a circuit.

Example. If the function of a circuit is described by a differential equation

$$a\frac{d^2V}{dt^2} + b\frac{dV}{dt} + cV = \delta(t),$$

where a, b, and c are constants, we can apply Laplace transform to the two sides of this equation to obtain

$$as^2 L_s[V] + bsL_s[V] + cL_s[V] = 1$$

By solving this for $L_s[V]$ one gets the transfer function of this circuit in *s*-domain.

$$H(s) \equiv L_s[V] = \frac{1}{as^2 + bs + c}$$

If the right hand side of this equation is not a δ-function but an arbitrary input voltage $V_{in}(t)$ then the response of the circuit will be

$$L_s[V] = \frac{L_s[V_{in}]}{as^2 + bs + c} = L_s[V_{in}]H(s)$$

Stability of a circuit

The circuit is stable if its response to a δ-function is finite. To find out if a circuit is stable we have to apply the inverse Laplace transform to its transfer function. If the transfers function is a ratio of two polynoms, this can be done using the partial fraction expansion method. Because the inverted Laplace transform of

$$\frac{c + j\cdot d}{s - (a + j\cdot b)}$$

is

$$(c + j\cdot b)\exp\left[(a + j\cdot b)t\right],$$

<u>the circuit is stable if real parts of all roots of the denominator in the ratio of the two polynoms are *negative*</u>.

Roots of the numerator are called zeros of the transfer function.

In the frequency domain, the transfer function is defined as the ratio of the output complex amplitude to the input complex amplitude if the input is given by $V_{in}\exp(j\omega t)$. If we substitute $j\omega$ instead of s in the expression for the transfer function s-domain, we will get the transfer function in ω-domain.

Example.

$$a\frac{d^2V}{dt^2}+b\frac{dV}{dt}+cV=V_{in}\exp(j\omega t)$$

We seek a solution in the form of $V=V_{out}\exp(j\omega t)$. Substitution leads to

$$a(j\omega)^2V_{out}+j\omega bV_{out}+cV_{out}=V_{in}$$

And one gets

$$H(\omega)=\frac{V_{out}}{V_{in}}=\frac{1}{a(j\omega)^2+j\omega b+c}.$$

Compare this to the above equation for $H(s)$.

Fourier transformation	**Laplace transformation**
$f(t)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}F(\omega)\exp(j\omega t)d\omega$	$f(t)=\frac{1}{2\pi j}\int\limits_{\gamma-j\infty}^{\gamma+j\infty}L_s[f]\exp(st)ds$ The constant γ is chosen such that all the singularities of $L_s[f]$ are on the left form the vertical line $s=\gamma$ on the complex plane.
$F_\omega[f]=\int\limits_{-\infty}^{\infty}f(t)\exp(-j\omega t)dt$	$L_s[f]=\int\limits_{0}^{\infty}f(t)\exp(-st)dt$
$F_\omega[f']=j\omega F_\omega[f]$ $F_\omega[f'']=(j\omega)^2F_\omega[f]$	$L_s[f']=-f(0)+sL_s[f]$ $L_s[f'']=-f'(0)-sf(0)+s^2L_s[f]$
	$L_s^{-1}\left[\frac{L_s[f]}{s}\right]=\int_0^t f(\tau)d\tau$
	$L[g*f]=L\left[f(t)\right]\cdot L\left[g(t)\right]$
$\frac{dF_\omega[f]}{d\omega}=-jF_\omega(t\cdot f)$	$\frac{dL_s[f]}{ds}=-L_s[t\cdot f]$

Table of Laplace transforms

Function $f(t)$	Laplace transform
1	$\frac{1}{s},\ s>0$
$2e^{-at}\left[c\cdot\cos(\omega t)+d\cdot\sin(\omega t)\right]$	$\frac{c-jd}{s+a-j\omega}+\frac{c+jd}{s+a+j\omega}$
$\exp(at)$	$\frac{1}{s-a},\ s>a$
t^n	$\frac{n!}{s^{n+1}},\ s>0$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2},\ s>0$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2},\ s>a$
$\exp(at)\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2},\ s>a$
$\exp(at)\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2},\ s>a$
$t\cos(\omega t)$	$\frac{s-\omega^2}{\left(s^2+\omega^2\right)^2},\ s>0$
$t\sin(\omega t)$	$\frac{2s\omega}{\left(s^2+\omega^2\right)^2},\ s>0$
$\delta(t)$	$L_s[\delta]=1$