Simulating many-body physics with quantum phase-space methods

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ACQAO Theory @ UQ

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Simplicity of Photons and Ultracold Gases

✔ underlying interactions are well understood

✔ easily characterised by a few parameters

✔ interactions can be tuned

$\longrightarrow$ use simple theoretical models to high accuracy

$\longrightarrow$ develop and test new methods of calculation

Theoretical Methods

▷ deterministic methods:

- ➜ exact diagonalisation ✘ intractable for $\gtrsim 5$ particles
- ➜ factorization ✘ not applicable for strong correlations
- ➜ perturbation theory ✘ diverges at strong couplings

▷ probabilistic methods:

- ➜ quantum Monte Carlo (QMC)
- ➜ stochastic wavefunction/fields
- ➜ phase-space methods

Overview

▷ introduction to phase-space representations

▷ density operator description of quantum evolution (3 classes)

➜ static, unitary and open

▷ Gaussian operator bases (3 types)

➜ coherent, thermal and squeezed

▷ applications (3 examples)

➜ pulse propagation in optical fibres *(photons)*
➜ Hubbard model *(atoms)*
➜ simple atomic-molecular dynamics *(molecules)*

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Phase-space distributions

▷ A classical state can be represented by a joint probability distribution in phase space $P(\mathbf{x},\mathbf{p})$

▷ 1932: Wigner constructed an analogous quantity for a quantum state:

$$W(x,p) = \frac{2}{\pi}\int dy \psi^*(x-y)\psi(x+y)\exp(-2iyp/\hbar)$$

✔ Wigner function gives correct marginals: $\int dx W(x,p) = 2\hbar P(p)$
$\int dy W(x,p) = 2\hbar P(x)$

✘ but it is not always positive $\longrightarrow$ not a true joint probability

▷ a positive Wigner function is a hidden variable theory

Probability distributions

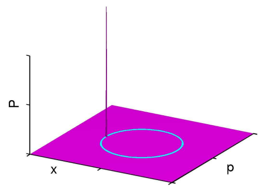
▷ many ways to define phase-space distributions:

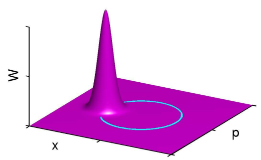
➜ eg Wigner, Husimi Q and Glauber-Sudarshan P

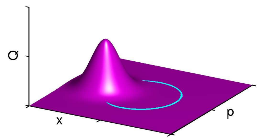
- all defined in terms of coherent states
- correspond to different choices of orderings

▷ to be a probabilistic representation, the phase-space functions must:

	P	W	Q
exist and be nonsingular	✗	✔	✔
always be positive	✗	✗	✔
evolve via drift and diffusion	✗	✗	✗







Reversibility

▷ classical random process is irreversible

➜ outward (positive) diffusion

▷ quantum mechanics is reversible

➜ phase-space functions generally don't have positive diffusion

A solution!

▷ dimension doubling

➜ diffusion into 'imaginary' dimensions ✔

➜ observables evolve reversibly ✔

➜ also fixes up existence and positivity ✔

Phase-space representation

$$\hat{\rho} = \int P(\vec{\lambda})\hat{\Lambda}(\vec{\lambda})d\vec{\lambda}$$

- ▷ $P(\vec{\lambda})$is a probability distribution
- ▷ $\hat{\Lambda}(\vec{\lambda})$ is a suitable operator basis
- ▷ $\vec{\lambda}$ is a generalised phase-space coordinate
- ▷ $d\vec{\lambda}$ is an integration measure
- ▷ equivalent to

$$\hat{\rho} = E\left[\hat{\Lambda}(\vec{\lambda})\right]$$

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- ▷ Gaussian operator bases (3 types)
  - ➜ coherent, thermal and squeezed
- ▷ applications (3 examples)
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Density operators for quantum evolution

1. Unitary dynamics: $\widehat{\rho}(t) = e^{-i\widehat{H}t/\hbar}\widehat{\rho}(0)e^{i\widehat{H}t/\hbar}$

   ▷ $\frac{\partial}{\partial t}\widehat{\rho} = -\frac{i}{\hbar}\left[\widehat{H},\widehat{\rho}\right]$

2. Equilibrium state: $\widehat{\rho}_{\mathrm{un}}(T) = e^{-\left(\widehat{H}-\mu\widehat{N}\right)/k_B T}$

   ▷ $\frac{\partial}{\partial\beta}\widehat{\rho} = \frac{1}{2}\left[\widehat{H}-\mu\widehat{N},\widehat{\rho}\right]_+ ;\ \beta = 1/k_B T$

3. Open dynamics: $\widehat{\rho}_{\mathrm{Sys}} = \mathrm{Tr}_{\mathrm{Res}}\{\widehat{\rho}\}$

   ▷ $\frac{\partial}{\partial t}\widehat{\rho} = -\frac{i}{\hbar}\left[\widehat{H},\widehat{\rho}\right] + \gamma\left(2\widehat{R}\widehat{\rho}\widehat{R}^\dagger - \widehat{R}^\dagger\widehat{R}\widehat{\rho} - \widehat{\rho}\widehat{R}^\dagger\widehat{R}\right)$

▷ each type is equivalent to a Liouville equation for $\widehat{\rho}$:

$$\frac{d}{d\tau}\widehat{\rho} \;=\; \widehat{L}\left[\widehat{\rho}\right];\ \tau = t,\beta$$

Phase-space Recipe

1. **Formulate**: $\partial\widehat{\rho}/\partial\tau = \widehat{L}[\widehat{\rho}]$

2. **Expand**: $\int \partial P/\partial\tau \widehat{\Lambda} d\overrightarrow{\lambda} = \int P \widehat{L}\left[\widehat{\Lambda}\right] d\overrightarrow{\lambda}$

3. **Transform**: $\widehat{L}\left[\widehat{\Lambda}\right] = \mathcal{L}\widehat{\Lambda}$

4. **Integrate** by parts: $\int P \mathcal{L}\widehat{\Lambda} d\overrightarrow{\lambda} \Longrightarrow \int \widehat{\Lambda} \mathcal{L}' P d\overrightarrow{\lambda}$

5. **Obtain** Fokker-Planck equation: $\partial P/\partial\tau = \mathcal{L}' P$

6. **Sample** with stochastic equations for $\overrightarrow{\lambda}$

Stochastic Gauges

- ▷ Mapping from Hilbert space to phase space not unique

  ➜ many "gauge" choices

- ▷ Can alter noise terms B_{ij}, introduce arbitrary drift functions $g_j(\vec{\lambda})$

Weight $d\Omega/d\tau = \Omega[U + g_j\zeta_j]$

Trajectory $d\lambda_i/\partial\tau = A_i + B_{ij}[\zeta_j - g_j]$

- ▷ Can also choose different bases, identities

Interacting many-body physics

$$\hat{\rho} \Longrightarrow \overrightarrow{\lambda}$$

- ✔ many-body problems map to nonlinear stochastic equations
- ✔ calculations can be from first principles
- ✔ precision limited only by sampling error
- ✔ choose basis to suit the problem

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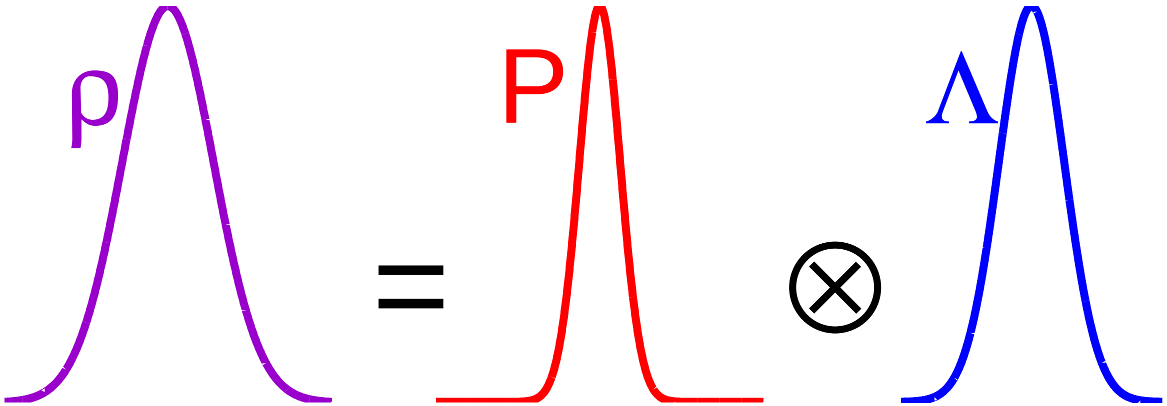
Operator Bases

- need basis simple enough to fit into a computer, complex enough to contain the relevant physics:

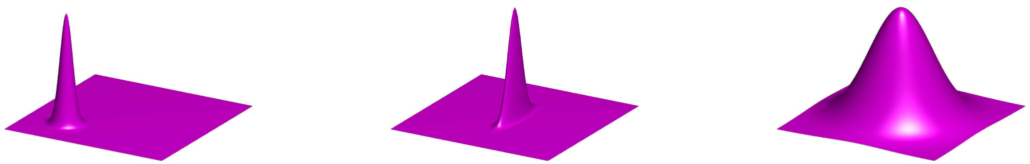


$$\sigma_\rho \sim \sigma_P + \sigma_\Lambda$$

General Gaussian operators

a generalisation of the density operators that describe Gaussian states

- ▷ Gaussian states can be:
  - ➜ coherent (for bosons), squeezed, or thermal



  - ➜ or any combination of these

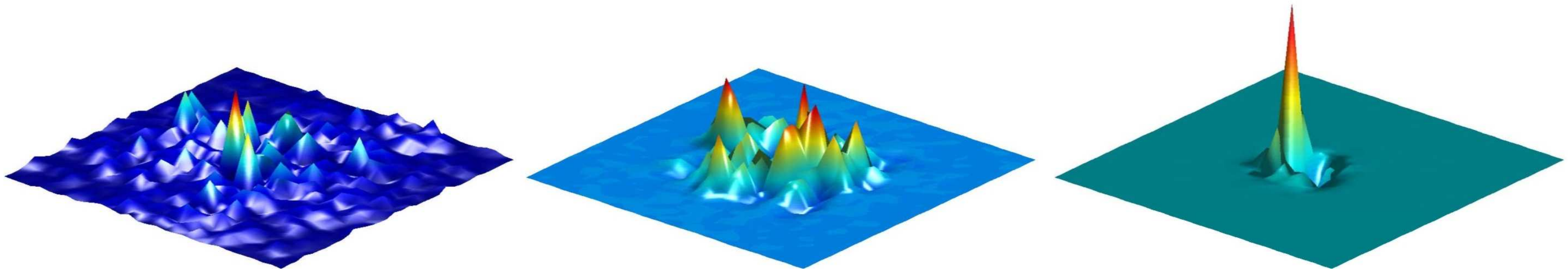
- ▷ characterised by first-order moments: $\overline{x}$, $\overline{p}$, $\overline{x^2}$, $\overline{p^2}$, $\overline{xp}$
  - ➜ all higher-order moments factorise

Gaussian Basis I: Coherent-state projectors

$$\widehat{\Lambda} = \frac{|\alpha\rangle\langle(\alpha^+)^*|}{\langle(\alpha^+)^*||\alpha\rangle}$$

- ▷ defines the $+P$ distribution, with a doubled phase space $\overrightarrow{\lambda} = (\Omega, \alpha, \alpha^+)$
- ▷ moments: $\left\langle O\left(\widehat{\mathbf{a}}^\dagger, \widehat{\mathbf{a}}\right)\right\rangle = E\left[O\left(\alpha^+, \alpha\right)\right]$
- ▷ successful for many applications in quantum optics
- ▷ successful simulations of short-time quantum dynamics of BEC

Evaporative Cooling of a BEC



▷ first-principles 3D calculation

- ➜ start with Bose gas above T_c; finish with narrow BEC peak
- ➜ 20000 atoms, 32000 modes
- ➜ Hilbert space is astronomically large

✘ Problems!

- ✘ method pushed to the limit
- ✘ breaks down for longer times, stronger interactions

Gaussian Basis II: Thermal operators

$$\widehat{\Lambda} = \left|\mathbf{I} \pm \mathbf{n}\right|^{\mp 1} : \exp\left[\widehat{\mathbf{a}}\left(\mathbf{I} \mp \mathbf{I} - [\mathbf{I} \pm \mathbf{n}]^{-1}\right)\widehat{\mathbf{a}}^{\dagger}\right] :$$

- ▷ now have a phase space of variances: $\overrightarrow{\lambda} = (\boldsymbol{\Omega}, \mathbf{n})$
- ▷ defined for bosons (upper sign) and *fermions* (lower sign)
- ▷ moments: $\left\langle \widehat{a}_i^{\dagger}\widehat{a}_j \right\rangle = E\left[n_{ij}\right]$, $\left\langle \widehat{a}_i^{\dagger}\widehat{a}_j^{\dagger}\widehat{a}_j\widehat{a}_i \right\rangle = E\left[n_{ii}n_{jj} \pm n_{ij}n_{ji}\right]$
- ▷ suitable for cold atoms

Gaussian Basis III: General form (including squeezing)

$$\widehat{\Lambda}(\overrightarrow{\lambda}) = \Omega\sqrt{\left|\underline{\underline{\sigma}}\right|}^{\mp 1} : \exp\left[\delta\underline{\widehat{a}}^{\dagger}\left(\underline{\underline{I}} \mp \underline{\underline{I}} - \underline{\underline{\sigma}}^{-1}\right)\delta\underline{\widehat{a}}/2\right] :$$

relative displacement: $\delta\underline{\widehat{a}} = \underline{\widehat{a}} - \underline{\alpha}$

annihilation and creation operators: $\underline{\widehat{a}} = \left(\widehat{a}_1, \ldots, \widehat{a}_M, \widehat{a}_1^{\dagger}, \ldots, \widehat{a}_M^{\dagger}\right)$

coherent offset: $\underline{\alpha} = \left(\alpha_1, \ldots, \alpha_M, \alpha_1^{+}, \ldots, \alpha_M^{+}\right)$, ($\underline{\alpha} = 0$ for fermions)

covariance: $\underline{\underline{\sigma}} = \begin{bmatrix} \mathbf{n}^T \pm \mathbf{I} & \mathbf{m} \\ \mathbf{m}^{+} & \mathbf{I} \pm \mathbf{n} \end{bmatrix}, \underline{\underline{I}} = \begin{bmatrix} \pm\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}.$

upper signs: bosons; **lower signs:** fermions

Extended phase space

$$\overrightarrow{\lambda} = (\boldsymbol{\Omega}, \boldsymbol{\alpha}, \boldsymbol{\alpha}^{+}, \mathbf{n}, \mathbf{m}, \mathbf{m}^{+})$$

$\Longrightarrow$ Hilbert-space dimension: 2^M for fermions, N^M for bosons

$\Longrightarrow$ phase-space dimension: $2(1-M+2M^2)$ for fermions, $2(1+3M+2M^2)$ for bosons

▷ Moments:

$$\begin{aligned} \langle \widehat{a}_i \rangle &= E[\alpha_i] \\ \left\langle \widehat{a}_i^\dagger \widehat{a}_j \right\rangle &= E\left[\alpha_i^+ \alpha_j + n_{ij}\right] \\ \langle \widehat{a}_i \widehat{a}_j \rangle &= E[\alpha_i \alpha_j + m_{ij}] \end{aligned}$$

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Application I: photons in a fibre

$$\widehat{H} = \widehat{H}_F + \widehat{H}_L + \widehat{H}_G + \widehat{H}_R$$

- ▷ $\widehat{H}_F$:fibre-optic Hamiltonian, including $\chi^{(3)}$nonlinearity
- ▷ $\widehat{H}_L$, $\widehat{H}_G$: coupling to absorbing reserviors and fibre amplifier reserviors
- ▷ $\widehat{H}_R$: nonlinear coupling to non-Markovian phonon reserviors
  ➜ models Raman transitions and Brillouin effect (GAWBS)
- ▷ have 10^2 modes and 10^9 particles

Scaled quantum field

▷ define a quantum photon-density field in terms of mode operators:

$$\widehat{\Psi}(t,x) = \frac{1}{\sqrt{2\pi}} \int dk\, \widehat{a}(t,k) e^{i(k-k_0)x + i\omega_0 t}\, ; \;\; \left[\widehat{\Psi}(t,x), \widehat{\Psi}^\dagger(t,x')\right] = \delta(x-x')$$

▷ change to propagative reference frame with scaled variables:

$t \Leftrightarrow (t - x/v)/t_0$	$x \Leftrightarrow x/x_0$	$\widehat{\phi} = \widehat{\Psi}\sqrt{vt_0/\overline{n}}$

➜ t_0 is a typical pulse duration

➜ $x_0 = t_0^2/|k''|$ is the dispersion length

➜ $\overline{n} = |k''|Ac/(n_2 \hbar \omega_c^2 t_0)$ is a typical photon number

Quantum Langevin Equations

- Raman-modified Heisenberg equations for photon-flux field:

$$\frac{\partial}{\partial x}\widehat{\phi}(t,x) = -\int_{-\infty}^{\infty} dt' g(t-t')\widehat{\phi}(t',x) + \widehat{\Gamma}(t,x) \pm \frac{i}{2}\frac{\partial^2}{\partial t^2}\widehat{\phi}(t,x)$$
$$+\left[i\int_{-\infty}^{\infty} dt' h(t-t')\widehat{\phi}^{\dagger}(t',x)\widehat{\phi}(t',x) + \widehat{\Gamma}^{R}(t,x)\right]\widehat{\phi}(t,x)$$

- correlations of the reservoir fields:

$$\left\langle \widehat{\Gamma}(\omega,x)\widehat{\Gamma}^{\dagger}(\omega',x')\right\rangle = \frac{\alpha^A}{\overline{n}}(\omega,x)\delta(x-x')\delta(\omega-\omega')$$
$$\left\langle \widehat{\Gamma}^{\dagger}(\omega',x')\widehat{\Gamma}(\omega,x)\right\rangle = \frac{\alpha^G}{\overline{n}}(\omega,x)\delta(x-x')\delta(\omega-\omega')$$
$$\left\langle \widehat{\Gamma}^{R\dagger}(\omega',x')\widehat{\Gamma}^{R}(\omega,x)\right\rangle = \frac{\alpha^R}{\overline{n}}(|\omega|)\left[n_{\mathrm{th}}(|\omega|)+\Theta(-\omega)\right]\delta(x-x')\delta(\omega-\omega')$$

Phase-Space Equations

▷ apply the phase-space recipe, use coherent-state basis

▷ two choices:

1. ***+P***

   (a) exact
   (b) defined on a doubled phase space
   (c) maps to normally ordered correlations

2. **Wigner**

   (a) approximation, good for large mode occupations, short times
   (b) defined on a classical phase space
   (c) maps to symmetrically ordered correlations

Wigner Equations

▷ get stochastic, Raman-modified nonlinear Schrödinger equation:

$$\begin{aligned}\frac{\partial}{\partial x}\phi(t,x) &= -\int_{-\infty}^{\infty} dt' g(t-t')\phi(t',x) + \Gamma(t,x) \pm \frac{i}{2}\frac{\partial^2}{\partial t^2}\phi(t,x) \\ &+ \left[i\int_{-\infty}^{\infty} dt' h(t-t')\phi^*(t',x)\phi(t',x) + \Gamma^R(t,x)\right]\phi(t,x)\end{aligned}$$

▷ noise correlations:

$$\begin{aligned}\langle \Gamma(\omega,x)\Gamma^*(\omega',x')\rangle &= \frac{\alpha^A(\omega)+\alpha^G(\omega)}{2\overline{n}}\delta(x-x')\delta(\omega-\omega') \\ \langle \Gamma^R(\omega,x)\Gamma^{R*}(\omega',x')\rangle &= \frac{\alpha^R}{\overline{n}}(|\omega|)\left[n_{\rm th}(|\omega|)+\frac{1}{2}\right]\delta(x-x')\delta(\omega-\omega') \\ \langle \Delta\phi(t,0)\,\Delta\phi^*(t',0)\rangle &= \frac{1}{2\overline{n}}\delta(t-t')\end{aligned}$$

+P Equations

- get two stochastic Raman-modified nonlinear Schrödinger equations:

$$
\begin{aligned}
\frac{\partial}{\partial x}\phi(t,x) &= -\int_{-\infty}^{\infty} dt' g(t-t')\phi(t',x) + \Gamma(t,x) \pm \frac{i}{2}\frac{\partial^2}{\partial t^2}\phi \\
&\quad + \left[i\int_{-\infty}^{\infty} dt' h(t-t')\phi^{+}(t',x)\phi(t',x) + \Gamma^{R}(t,x) \right] \phi(t,x) \\
\frac{\partial}{\partial x}\phi^{+}(t,x) &= -\int_{-\infty}^{\infty} dt' g^{*}(t-t')\phi^{+}(t',x) + \Gamma^{+}(t,x) \mp \frac{i}{2}\frac{\partial^2}{\partial t^2}\phi \\
&\quad + \left[-i\int_{-\infty}^{\infty} dt' h^{*}(t-t')\phi(t',x)\phi^{+}(t',x) + \Gamma^{R+}(t,x) \right] \phi^{+}(t,x)
\end{aligned}
$$

- for non-classical states, ϕ and ϕ^{+} are *not* complex conjugate

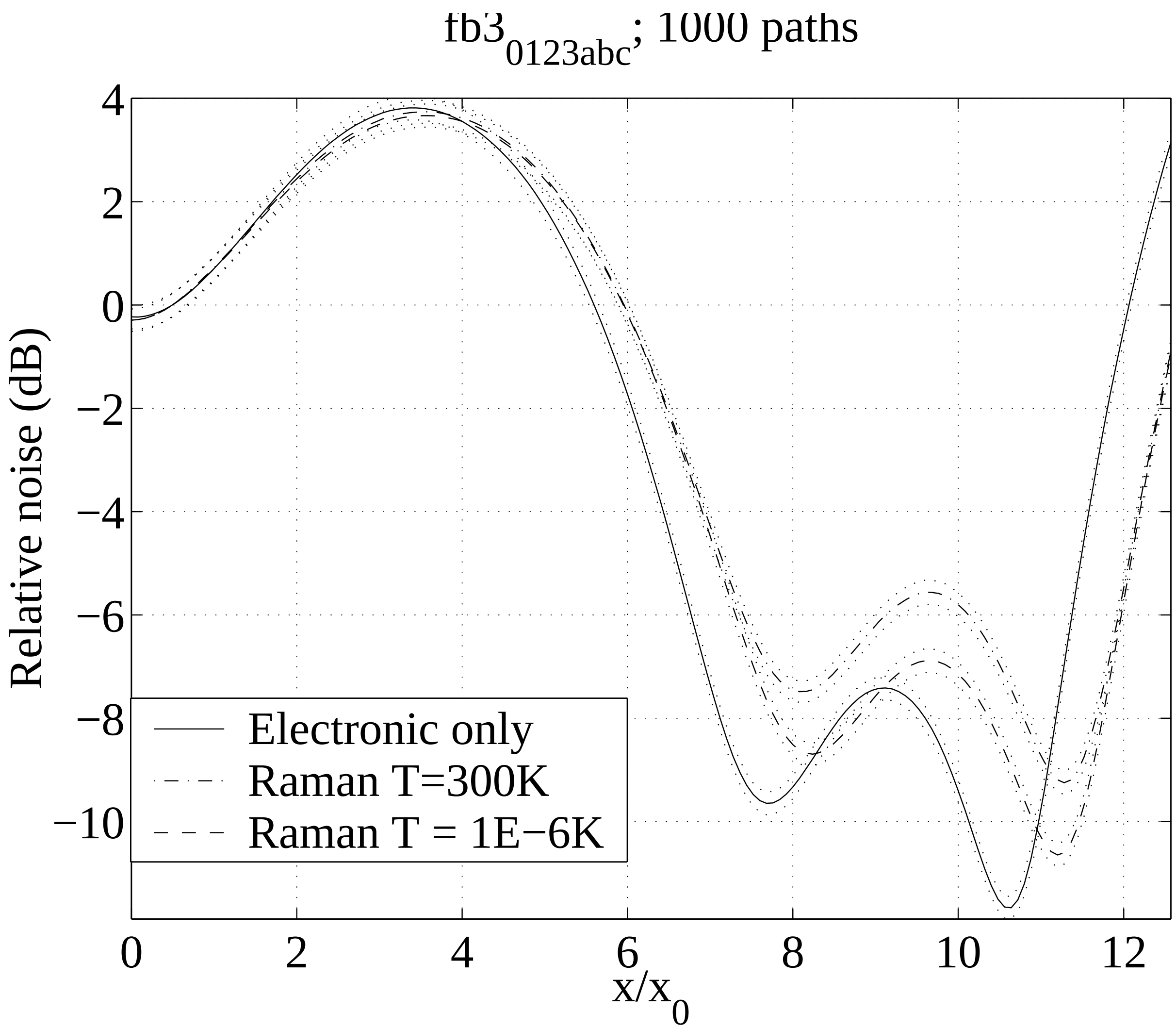
+P noise correlations

$$
\begin{aligned}
\langle \Gamma(\omega,x)\Gamma^*(\omega',x')\rangle &= \frac{\alpha^G(\omega)}{\bar{n}}\delta(x-x')\delta(\omega-\omega') \\
\left\langle \Gamma^R(\omega,x)\Gamma^{R+}(\omega',x')\right\rangle &= \frac{\alpha^R}{\bar{n}}(|\omega|)\left[n_{\mathrm{th}}(|\omega|)+\Theta(-\omega)\right]\delta(x-x')\delta(\omega-\omega') \\
\left\langle \Gamma^R(\omega,x)\Gamma^R(\omega',x')\right\rangle &= \frac{1}{\bar{n}}\left\{\alpha^R(|\omega|)\left[n_{\mathrm{th}}(|\omega|)+\Theta(-\omega)\right]-i\mathrm{Re}\left[h(\omega)\right]\right\} \\
&\quad \times\delta(x-x')\delta(\omega+\omega')
\end{aligned}
$$

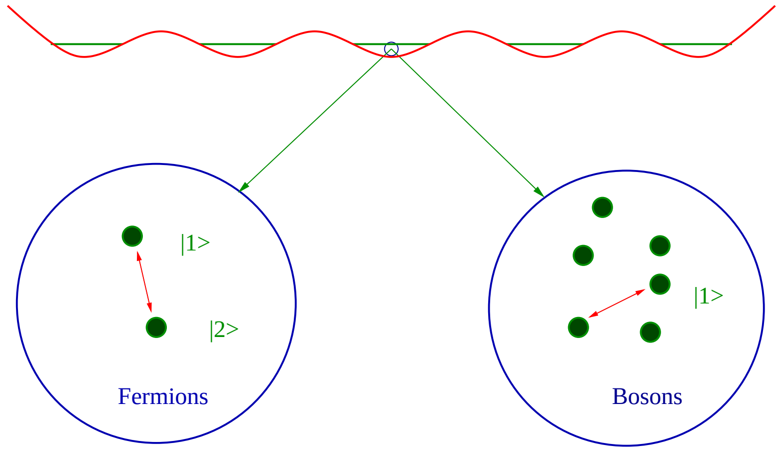
- ▷ no initial noise for a coherent state
- ▷ but there is multiplicative noise due to spontaneous scattering

Simulations

▷ soliton jitter, soliton squeezing, supercontinuum generation



Application II: atoms in a lattice



$$\widehat{H} = -\sum_{ij,\sigma} t_{ij}\widehat{c}^{\dagger}_{i,\sigma}\widehat{c}_{j,\sigma} + U\sum_{j}\widehat{c}^{\dagger}_{j,\uparrow}\widehat{c}^{\dagger}_{j,\downarrow}\widehat{c}_{j,\downarrow}\widehat{c}_{j,\uparrow}$$

▷ simplest model of an interacting Fermi gas on a lattice

➜ weak-coupling limit $\longrightarrow$ BCS transitions

➜ solid-state models; relevance to High-T_c superconductors

Solving the Hubbard Model

▷ only the 1D model is exactly solvable (Lieb & Wu, 1968)

▷ even then, not all correlations can be calculated

▷ higher dimensions - can use Quantum Monte Carlo methods.

✘ except for a few special symmetrical cases, QMC suffers from sign problems with the Hubbard model

  ▷ e.g. sign problems for repulsive interaction away from half filling

✘ sign problem increases with dimension, lattice size, interaction strength

Fermionic sign problem

- ▷ Quantum Monte Carlo (QMC) samples many-body wavefunction $\phi(r)$ (wave-function treated as a probability)

- ▷ but Fermion states are antisymmetric

  ➜ wavefunction nonpositive

- ▷ must introduce (possibly negative) weighting factors

  ➜ bad sampling errors (unless approximations used)

$$\langle A \rangle \sim \frac{\langle sA \rangle}{\langle s \rangle}$$

Applying the Gaussian representation

▷ Use thermal basis, and apply mappings

$$\widehat{\mathbf{n}}_\sigma \widehat{\rho} \quad\rightarrow\quad \left\{2\mathbf{n}_\sigma - (\mathbf{I}-\mathbf{n}_\sigma)\frac{\partial}{\partial \mathbf{n}_\sigma}\mathbf{n}_\sigma\right\} P(\Omega,\mathbf{n}_\uparrow,\mathbf{n}_\downarrow)$$

$$\widehat{\rho}\widehat{\mathbf{n}}_\sigma \quad\rightarrow\quad \left\{2\mathbf{n}_\sigma - \mathbf{I} - \mathbf{n}_\sigma\frac{\partial}{\partial \mathbf{n}_\sigma}(\mathbf{I}-\mathbf{n}_\sigma)\right\} P(\Omega,\mathbf{n}_\uparrow,\mathbf{n}_\downarrow)$$

$$\widehat{\rho} \quad\rightarrow\quad -\frac{\partial}{\partial \Omega}\Omega P(\Omega,\mathbf{n}_\uparrow,\mathbf{n}_\downarrow)$$

$\Longrightarrow$ Fokker-Planck equation for P, with drift and diffusion

$\Longrightarrow$ sample with stochastic equations for Ω and $\mathbf{n}_\sigma$

Positive-Definite Diffusion

▷ Modify interaction term with a 'Fermi gauge':

$$U\sum_j :\widehat{n}_{jj,\downarrow}\widehat{n}_{jj,\uparrow}: \;=\; -\frac{1}{2}|U|\sum_j :\left(\widehat{n}_{jj,\downarrow} - \frac{U}{|U|}\widehat{n}_{jj,\uparrow}\right)^2:$$

$\Longrightarrow$ diffusion matrix has a real 'square root' matrix

- $\Longrightarrow$ realise the diffusion with a real noise process
- $\Longrightarrow$ problem maps to a real (and much more stable) subspace

Stratonovich Equations

▷ Itô stochastic equations, in matrix form:

$$\frac{d\Omega}{d\tau} = -\Omega\left\{-\sum_{ij,\sigma} t_{ij} n_{ij,\sigma} + U\sum_{j} n_{jj,\downarrow} n_{jj,\uparrow} - \mu\sum_{j,\sigma} n_{jj,\sigma}\right\}$$

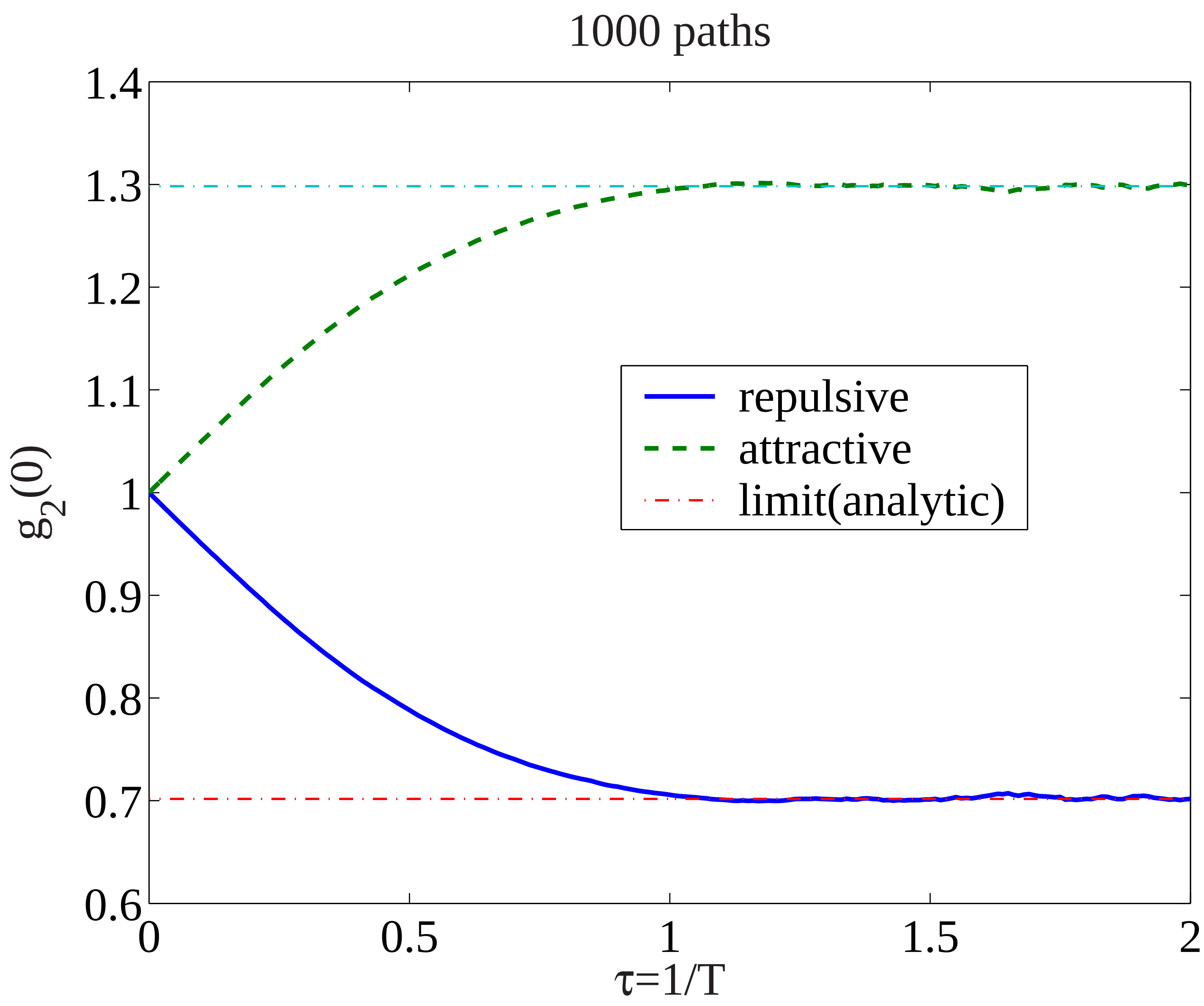
$$\frac{d\mathbf{n}_\sigma}{d\tau} = -\frac{1}{2}\left\{\left(\mathbf{I}-\mathbf{n}_\sigma\right)\Delta_\sigma^{(1)}\mathbf{n}_\sigma + \mathbf{n}_\sigma\Delta_\sigma^{(2)}\left(\mathbf{I}-\mathbf{n}_\sigma\right)\right\},$$

where the stochastic propagator matrix is

$$\Delta_{ij,\sigma}^{(r)} = \left[-t_{ij} + \delta_{ij}\left\{U n_{jj,\sigma'} - \mu\right\}\right] \pm \delta_{ij}\sqrt{2|U|}\xi_j^{(r)}$$

▷ $\xi_j^{(r)}$ are delta-correlated white noises

1D Lattice-100 sites



Branching

▷ averages are weighted,eg

$$\langle \widehat{\mathbf{n}}(\tau) \rangle = \frac{\sum_{j=1}^{N_p} \mathbf{\Omega}^{(j)}(\tau)\mathbf{n}^{(j)}(\tau)}{\sum_{j=1}^{N_p} \mathbf{\Omega}^{(j)}(\tau)}$$

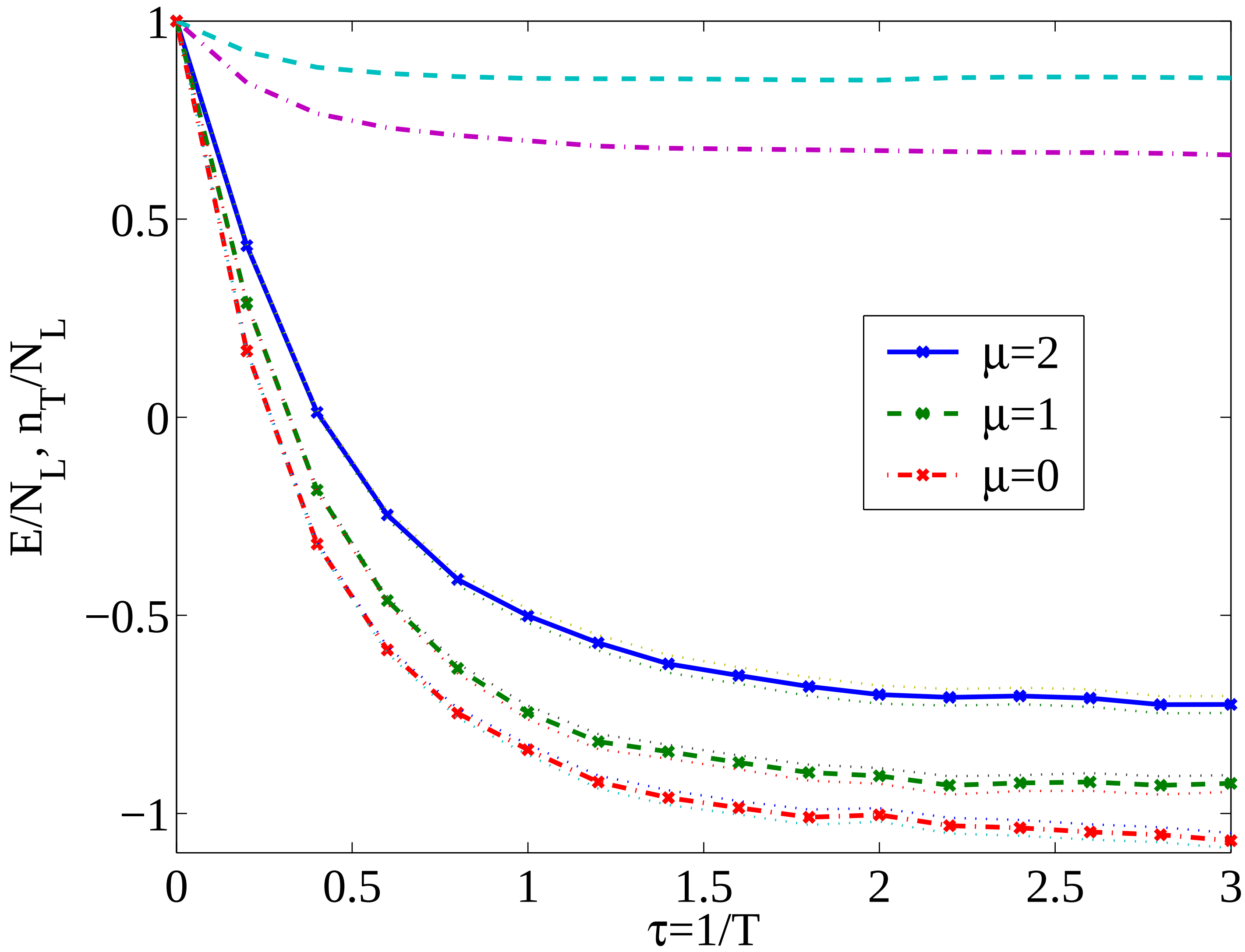
✘ but weights spread exponentially $\Longrightarrow$ many irrelevant paths

$\Longrightarrow$ delete low-weight paths and clone high-weight paths:

$$m^{(jp)} = \text{Integer}\left[\xi + \mathbf{\Omega}^{(jp)}/\overline{\mathbf{\Omega}}\right]$$

- ▷ $\xi \in [0,1]$ is a random variable, $\overline{\Omega}$ is an average weight
- ▷ after branching, weights of surviving paths are equalised

16x16 2D Lattice



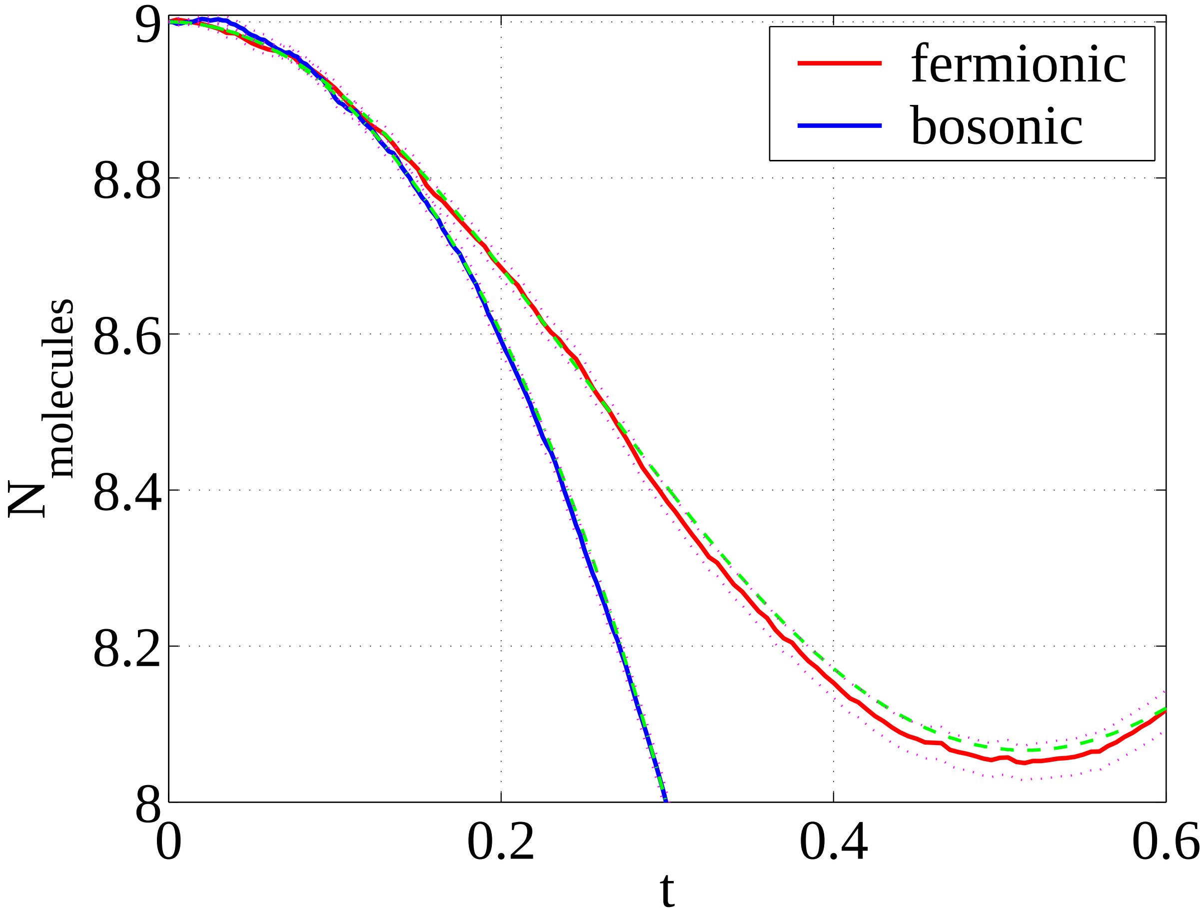
No sign problem!

Application III: Molecules in a well

- Hamiltonian: $\widehat{H} = \widehat{a}\widehat{b}_1^\dagger\widehat{b}_2^\dagger + \widehat{a}^\dagger\widehat{b}_1\widehat{b}_2$

$$\begin{aligned}
\dot{n}_1 &= i\chi(\alpha^+ m - \alpha m^+) \pm \sqrt{i\chi} n_1 \left(m\zeta_1^* + m^+\zeta_2^*\right) \ , \\
\dot{n}_2 &= i\chi(\alpha^+ m - \alpha m^+) \pm \sqrt{i\chi} n_2 \left(m\zeta_1^* + m^+\zeta_2^*\right) \ , \\
\dot{m} &= -i\chi\alpha(1 \pm n_1 \pm n_2) + \sqrt{i\chi} \left(\pm m^2\zeta_1^* + n_1 n_2 \zeta_2^*\right) \ , \\
\dot{m}^+ &= i\chi\alpha^+(1 \pm n_1 \pm n_2) + \sqrt{i\chi} \left(n_1 n_2 \zeta_1^* \pm m^{+2}\zeta_2^*\right) \ , \\
\dot{\alpha} &= -i\chi m - \sqrt{i\chi}\zeta_1 \ , \\
\dot{\alpha}^+ &= i\chi m^+ + \sqrt{i\chi}\zeta_2 \ ,
\end{aligned}$$

Result: Pauli blocking



Summary

▷ Generalised phase-space representations provide a means of simulating many-body quantum physics from first principles, with *precision limited only by sampling error*.

▷ Coherent-state-based methods have been successful in simulating quantum dynamics of photons and weakly interacting ultracold gases.

▷ Gaussian-based methods extend the applicability to highly correlated systems of bosons and *fermions*.

▷ Simulated the Hubbard model (fermions in a lattice) *without sign errors*.