MATH4105 General Relativity Problem Sheet 5 Feedback

I. QUESTIONS

1/ *(Hartle problem 5.12) (a) Show explicitly that the straight line path between any two points in flat three-dimensional space ($dS^2 = dx^2 + dy^2 + dz^2$) is the shortest distance between them.*

(b) Is the straight line path between two spacelike separated points in flat spacetime the shortest distance between them?

(a) The solution to this is just as the analysis of the twin paradox discussed in literature.

Start by assuming that the two points are separated along the x-axis, so that the start point is $x = 0, y = 0, z = 0$ and the endpoint is $x = x_0, y = 0, z = 0$. Consider a general path connecting these two fields. In principle we could choose a general parameter to describe this path but we will use the parameter to be the value of x so that our path is $(x, y(x), z(x))$. We have

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{1}$$

or in more sensible notation

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2} \tag{2}$$

Now since dy/dx appears squared, the length gets sorter if it is equal to zero, the same is true for dz/dx. So we conclude that the length of the straight line path is the shortest possible

$$s = \int_0^{x_0} dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2} \geq \int_0^{x_0} dx = x_0. \tag{3}$$

Thus the geodesic straight line path is the shortest possible.

(b) Now this is almost the same. The first thing is to recall that we can choose the reference frame as we like and the length will be the same. Now with a spacelike separated set of points we can choose the reference frame so that the endpoints have the same time co-ordinate, lets call that $t = 0$. As before we can choose the endpoints to be on the x-axis. If we take the eminently reasonable view that sensible paths should have $dt/dx = 0$ then the calculation is exactly as before and we conclude that the straight line path is the shortest such path.

On the other hand consider the path $(t(x), x, 0, 0)$ with

$$t(x) = \begin{cases} vx & 0 \leq x \leq x_0/2 \\ v(x_0 - x) & x_0/2 \leq x \leq x_0 \end{cases} \tag{4}$$

Clearly, although this path makes no physical sense since it goes first forwards then backwards in time, it does go between the right endpoints. So long as $v < 1$ we have

$$g_{\mu\nu} \frac{dx^\mu}{dx} \frac{dx^\nu}{dx} = 1 - v^2 > 0 \tag{5}$$

so this is a spacelike path. (If we were using "proper length" as the parameter this quantity would have to be one, but in general if it is positive the path is spacelike.) Then we have

$$s = \int_0^{x_0} dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2 - \left(\frac{dt}{dx}\right)^2} = x_0\sqrt{1 - v^2}. \tag{6}$$

By choosing v as close as we like to one we can make the length of this path as small as we want. So it is clear that if we allow such paths that the geodesic path is actually like a point of inflection in that there are paths that are shorter and paths that are longer than the geodesic.

2/ *(Hartle problem 5.4) A satellite orbits the Earth in the same direction it rotates in a circular orbit above the equator a distance of 200 km from the surface. By how many seconds per day will a clock on such a satellite run slow compared to a clock on the Earth? (Compute just the special relativistic effects.)*

The satellite of mass m has a circular orbit of radius R with gravitational force equal to the centripetal acceleration

$$\frac{GMm}{R^2} = \frac{mv^2}{R} \tag{7}$$

where M is the mass of the sun. This gives us

$$|v| = \sqrt{\frac{GM}{R}}. \tag{8}$$

The satellite is above the equator and the velocity vectory is aligned along the same direction as the earth's rotation, we'll label this direction x so we have $u_{\text{sat}} = (\gamma, \gamma|v|, 0, 0)$ with $\gamma = (1 - |v|^2/c^2)^{-1/2}$ as usual.

A clock on the earth on the other hand is moving with a velocity

$$v_{\text{earth}} = \omega_{\text{earth}} r \tag{9}$$

where ω_{earth} is the angular frequency of the earth and r is the radius of the earth. So the velocity four-vector is $u_{\text{sat}} = (\gamma_{\text{earth}}, \gamma_{\text{earth}} v_{\text{earth}}, 0, 0)$.

The time elapsed on the clock on earth in a co-ordinate time t is

$$\tau_{\text{earth}} = \int_0^t dt \frac{d\tau}{dt} = \int_0^t dt/\gamma_{\text{earth}} = t/\gamma_{\text{earth}}. \tag{10}$$

Likewise the time elapsed on the satellite clock is

$$\tau_{\text{satellite}} = t/\gamma = \frac{\gamma_{\text{earth}}}{\gamma} \tau_{\text{earth}} \tag{11}$$

The satellite clock is running slower by an amount

$$\tau_{\text{earth}} - \tau_{\text{satellite}} = \left(1 - \frac{\gamma_{\text{earth}}}{\gamma}\right) \tau_{\text{earth}} \tag{12}$$

Putting in the numbers we have

$$v_{\text{earth}} = \frac{2\pi}{24 \times 60 \times 60} 6.378 \times 10^6 = 463.8 \text{ m/s}. \tag{13}$$

On the other hand the

$$|v| = \sqrt{\frac{(6.673 \times 10^{-11})(5.974 \times 10^{24})}{6.378 \times 10^6 + 2 \times 10^5}} = \sqrt{6.060 \times 10^7} = 7784 \text{ m/s}. \tag{14}$$

And so

$$\tau_{\text{earth}} - \tau_{\text{satellite}} = \left(1 - \frac{\gamma_{\text{earth}}}{\gamma}\right) \tau_{\text{earth}} \simeq \left[\left(\frac{|v|}{c}\right)^2 - \left(\frac{v_{\text{earth}}}{c}\right)^2\right] \tau_{\text{earth}} = 58\ \mu\text{s} \tag{15}$$

In the approximation I have just done a binomial expansion to lowest order, since my calculator didn't like calculating the γ factors.

3/ (Schutz problem 1.18) Use the special relativistic velocity addition law to show that if a particle has velocity $v = \tanh\theta$ *then in a frame moving at velocity* $V = \tanh\Theta$ *along the same direction the velocity of the particle is given by*

$$v' = \tanh(\theta + \Theta). \tag{16}$$

The question as written has a mistake in the sign of Θ but we didn't worry about this in the marking.

The velocity addition formula giving speed v' in new frame S' for particle with speed v in frame S when velocity of S' with respect to S is V is

$$v' = \frac{v - V}{1 - vV}. \tag{17}$$

The formula in the question has $V \to -V$ which describes the inverse operation, sorry about that but the sign doesn't matter really. If we now have $v = \tanh\theta$ and $V = \tanh\Theta$ when we obtain

$$\begin{aligned} v' &= \frac{\tanh\theta - \tanh\Theta}{1 - \tanh\theta\tanh\Theta} \\ &= \frac{\sinh\theta\cosh\Theta - \sinh\Theta\cosh\theta}{\cosh\theta\cosh\Theta - \sinh\Theta\sinh\theta} \\ &= \tanh(\theta - \Theta). \end{aligned}$$

We've used $\tanh\theta = \sinh\theta/\cosh\theta$ and found common denominators in the fractions in the second line. In the third line either look up the hyperbolic trig identities or use $\sinh\theta = [\exp\theta - \exp(-\theta)]/2$ and so on.

4/ (Hartle problem 5.20) Consider a particle with four-momentum $\mathbf{p}$ *and an observer with four-velocity* $\mathbf{u}$*. Show that if the particle goes through the observer's laboratory, the magnitude of the three-momentum measured is*

$$|\vec{p}| = \sqrt{(\mathbf{p}\cdot\mathbf{u})^2 + \mathbf{p}\cdot\mathbf{p}} \tag{18}$$

Suppose that we write the four-momentum in co-ordinates where the observer is at rest (that is we choose co-ordinates such that $\mathbf{u} = (1, 0, 0, 0)$) then $\mathbf{p} = (E, \vec{p})$ and we have

$$\mathbf{p}\cdot\mathbf{p} = |\vec{p}|^2 - E^2 = |\vec{p}|^2 - (\mathbf{p}\cdot\mathbf{u})^2 \tag{19}$$

Rearranging this we get the desired equation and since it is entirely in terms of dot products it is valid in all reference frames.