MATH4105 General Relativity Problem Sheet 6 Answers

1 *(Based on Hartle including problems 7.11,7.12) Consider a metric sometimes known as the Alcubierre warp drive. There is one of these metrics for any curve in space of the form $x_s(t), y = 0, z = 0$. We can define the velocity associated with the curve $v_s(t) = dx_s/dt$ and the distance of any spatial point from the curve $r_s(t) = (x - x_s(t))^2 + y^2 + z^2$. Finally we define a smooth positive function $f(r_s)$ such that $f(0) = 1$ and $f(r_s)$ vanishes when $r_s > R$ for some R. The line element specifying the metric is then*

$$ds^2 = -dt^2 + (dx - v_s f(r_s)dt)^2 + dy^2 + dz^2. \tag{1}$$

We will describe $x_s(t)$ as the path of a space ship undergoing a "warp-drive"

(a) Find the paths of light beams in this metric and show that at every point along $x_s(t)$ the four-velocity of the ship lies inside the forward light cone.

(b) How much ship time elapses on a trip between stations that takes co-ordinate time T?

(c) Show that the path $x^\mu(t) = (t, x_s(t), 0, 0)$ is a geodesic of this metric.

1a

We have the metric

$$ds^2 = -dt^2 + (dx - v_s f(r_s)dt)^2 + dy^2 + dz^2. \tag{2}$$

Null paths have $ds^2 = 0$. This gives

$$\left(\frac{dx}{dt} - v_s(t)f(r_s)\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = 1 \tag{3}$$

and for $dy = 0 = dz$ paths that we are interested in

$$\frac{dx}{dt} = \pm 1 + v_s f(r_s). \tag{4}$$

The forward light cones are tilted along the path $x_s(t)$ so that as we will see shortly this path is timelike. Far away from $x_s(t)$ they are vertical just as in flat spacetime.

1b

We're asked to consider the path

$$x^\mu(t) = (t, x_s(t), 0, 0). \tag{5}$$

Firstly let's check that this path is inside the forward light cones for this metric. For this path we have

$$\frac{dx}{dt} = v_s(t) \tag{6}$$

and light paths passing through the points on $x_s(t)$ and moving on the $t - x$ plane have the velocities $\pm 1 + v_s$ from the previous question. Finally since

$$-1 + v_s < v_s < 1 + v_s \tag{7}$$

the particle is moving at less than the local speed of light.

Alternatively we could check that the path is time-like all along its length. This is also sufficient. We have

$$\frac{dx^\mu(t)}{dt} = (1, v_s(t), 0, 0) \tag{8}$$

and

$$g_{\mu\nu}\frac{dx^\mu(t)}{dt}\frac{dx^\mu(t)}{dt} = -\left(\frac{dt}{dt}\right)^2 + \left(\frac{dx}{dt} - v_s(t)f(r_s)\frac{dt}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = -1 \tag{9}$$

so the path is timelike. (I've used $dt/dt = 1$, $dx/dt = v_s(t)$ and $r_s = 0$ all of which hold along the warp drive path.)

Notice that this implies that the proper time is $\tau = t$ since the proper time experienced by an observer moving along the curve $x^\mu(\sigma)$ is given by

$$\tau = \int_{\sigma_0}^{\sigma_1} \sqrt{-g_{\mu\nu}\frac{dx^\mu(\sigma)}{d\sigma}\frac{dx^\mu(\sigma)}{d\sigma}}d\sigma \tag{10}$$

and we have just substituted for our path into this formula to find

$$\tau = \int_{t_0}^{t_1} dt = \Delta t. \tag{11}$$

1c

Now we need to check that we have a geodesic. There are two ways of doing this. Firstly evaluate the Christoffel symbols along the path of interest using the expression for them in terms of the metric, or inspect the Euler-Lagrange equations directly.

If we set

$$w\left(x^\mu, \frac{dx^\mu}{d\tau}\right) = -g_{\mu\nu}\frac{dx^\mu(\tau)}{d\tau}\frac{dx^\mu(\tau)}{d\tau} = \left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dx}{d\tau} - v_s(t)f(r_s)\frac{dt}{d\tau}\right)^2 - \left(\frac{dy}{d\tau}\right)^2 - \left(\frac{dz}{d\tau}\right)^2 \tag{12}$$

then the Euler-Lagrange equations require

$$\frac{\partial w}{\partial x^\mu} = \frac{d}{d\tau}\frac{\partial w}{\partial(dx^\mu/d\tau)}. \tag{13}$$

Now we need to be careful. Firstly one needs to do the partial differentiations of w before substituting for the current path (remember we have to regard x^μ and $dx^\mu/d\tau$ as the independent variables when performing these derivatives). Secondly notice that r_s depends on time as well as position since

$$r_s = \sqrt{[x - x_s(t)]^2 + y^2 + z^2}. \tag{14}$$

So we have

$$\frac{\partial f(r_s)}{\partial t} = \frac{v_s f'(r_s)}{r_s}(x - x_s(t)) \tag{15}$$

$$\frac{\partial f(r_s)}{\partial x} = \frac{f'(r_s)}{r_s}(x - x_s(t)) \tag{16}$$

$$\frac{\partial f(r_s)}{\partial y} = \frac{f'(r_s)y}{r_s} \tag{17}$$

$$\frac{\partial f(r_s)}{\partial z} = \frac{f'(r_s)z}{r_s} \tag{18}$$

where $f' = df/dr_s$. Notice that for $x^\mu(\tau) = (\tau, x_s(\tau) = x_s(t), 0, 0)$, our path, all of these are zero.

Now consider

$$\frac{\partial w}{\partial t} = -2\left[v'f + v\frac{\partial f(r_s)}{\partial t}\right]\frac{dt}{d\tau}\left(\frac{dx}{d\tau} - v_s(t)f(r_s)\frac{dt}{d\tau}\right) \tag{19}$$

$$\frac{\partial w}{\partial x} = -2v\frac{\partial f(r_s)}{\partial x}\frac{dt}{d\tau}\left(\frac{dx}{d\tau} - v_s(t)f(r_s)\frac{dt}{d\tau}\right) \tag{20}$$

$$\frac{\partial w}{\partial y} = -2v\frac{\partial f(r_s)}{\partial y}\frac{dt}{d\tau}\left(\frac{dx}{d\tau} - v_s(t)f(r_s)\frac{dt}{d\tau}\right) \tag{21}$$

$$\frac{\partial w}{\partial z} = -2v\frac{\partial f(r_s)}{\partial z}\frac{dt}{d\tau}\left(\frac{dx}{d\tau} - v_s(t)f(r_s)\frac{dt}{d\tau}\right). \tag{22}$$

Notice that for our path all of these are zero.

Finally consider

$$\frac{\partial w}{\partial(dt/d\tau)} = 2\frac{dt}{d\tau} + 2v_s f\left(\frac{dx}{d\tau} - v_s(t)f(r_s)\frac{dt}{d\tau}\right) \tag{23}$$

$$\frac{\partial w}{\partial(dx/d\tau)} = -2\left(\frac{dx}{d\tau} - v_s(t)f(r_s)\frac{dt}{d\tau}\right) \tag{24}$$

$$\frac{\partial w}{\partial(dy/d\tau)} = -2\frac{dy}{d\tau} \tag{25}$$

$$\frac{\partial w}{\partial(dz/d\tau)} = -2\frac{dz}{d\tau} \tag{26}$$

Now for our path $dx^\mu/d\tau = (1, v_s(t), 0, 0)$ so these equations give us $2, 0, 0, 0$ respectively and so we have

$$\frac{d}{d\tau}\frac{\partial w}{\partial(dx^\mu/d\tau)} = 0 \tag{27}$$

for all μ and hence the geodesic equations are satisfied.

Notice that although I have to do the original differentiations before substituting for the path I then substitute as early as possible to get simple expressions.

Most people who attempted this did some version of what I have done here but the organisation can vary considerably, particularly if you use the formulae for Christoffel symbols in terms of the metric in which case one evaluates derivatives of the metric only not w.

2 *(Hartle 8.3) A three-dimensional spacetime has the line element*

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2 d\phi^2 \tag{28}$$

(a) Find the explicit Lagrangian for the variational principle for geodesics in this spacetime in these co-ordinates.
(b) Using the results of (a) write out the components of the geodesic equation by computing them from the Lagrangian.
(c) Read off the non-zero Christoffel symbols for this metric from your results in (b).

2a

Despite the fact that we are in three dimensions these formulae are exactly the same as in the usual case. We can parameterize a path $x^\mu(\lambda)$ and find the proper time associated with this path to be

$$\tau_{AB} = \int_A^B d\lambda\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^n u}{d\lambda}} = \int_A^B d\lambda\sqrt{w} \tag{29}$$

Geodesics will extremize this proper time and so the argument of this integral is the Lagrangian that we need to use. So we find in our case

$$L(x^\mu, dx^\mu/d\lambda) = \sqrt{\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 - \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\phi}{d\lambda}\right)^2} \tag{30}$$

The Euler-Lagrange equations will be unchanged if we use proper time τ as the parameter and take the Lagrangian to be the square of this. I discussed this in lectures and it's in the lecture notes so I won't go through the calculation here but I would also have accepted the following answer with some explanation

$$L'(x^\mu, dx^\mu/d\tau) = \left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\phi}{d\tau}\right)^2 \tag{31}$$

2b

Now the Euler-Lagrange equations are

$$\frac{d}{d\tau}\frac{\partial L'}{\partial(dx^\mu/d\tau)} = \frac{\partial L'}{\partial x^\mu} \tag{32}$$

where we have used the L' version to avoid the square roots. Look in the lecture notes for how to convert between these two.

There is nothing for it but to calculate the derivatives. I find

$$\frac{\partial L'}{\partial t} = 0 \tag{33}$$

$$\frac{\partial L'}{\partial r} = \frac{2M}{r^2}\left(\frac{dt}{d\tau}\right)^2 + \frac{2M}{r^2}\left(1-\frac{2M}{r}\right)^{-2}\left(\frac{dr}{d\tau}\right)^2 - 2r\left(\frac{d\phi}{d\tau}\right)^2 \tag{34}$$

$$\frac{\partial L'}{\partial \phi} = 0 \tag{35}$$

$$\tag{36}$$

and

$$\frac{\partial L'}{\partial(dt/d\tau)} = 2\left(1-\frac{2M}{r}\right)\frac{dt}{d\tau} \tag{37}$$

$$\frac{\partial L'}{\partial(dr/d\tau)} = -2\left(1-\frac{2M}{r}\right)^{-1}\frac{dr}{d\tau} \tag{38}$$

$$\frac{\partial L'}{\partial(d\phi/d\tau)} = -2r^2\frac{d\phi}{d\tau} \tag{39}$$

$$\tag{40}$$

When we evaluate $d/d\tau$ we need to use the product and the chain rule, we need to do partial derivatives with respect to all of the variables and their derivatives. We find

$$\frac{d}{d\tau}\frac{\partial L'}{\partial(dt/d\tau)} = 2\left(1-\frac{2M}{r}\right)\frac{d^2t}{d\tau^2} + \frac{4M}{r^2}\frac{dt}{d\tau}\frac{dr}{d\tau} \tag{41}$$

$$\frac{d}{d\tau}\frac{\partial L'}{\partial(dr/d\tau)} = -2\left(1-\frac{2M}{r}\right)^{-1}\frac{d^2r}{d\tau^2} + \frac{4M}{r^2}\left(1-\frac{2M}{r}\right)^{-2}\left(\frac{dr}{d\tau}\right)^2 \tag{42}$$

$$\frac{d}{d\tau}\frac{\partial L'}{\partial(d\phi/d\tau)} = -2r^2\frac{d^2\phi}{d\tau^2} - 4r\frac{dr}{d\tau}\frac{d\phi}{d\tau}. \tag{43}$$

$$\tag{44}$$

We can now rewrite the Euler-Lagrange equations to give us the accelerations

$$\frac{d^2t}{d\tau^2} = -\frac{2M}{r^2}\left(1-\frac{2M}{r}\right)^{-1}\frac{dt}{d\tau}\frac{dr}{d\tau} \tag{45}$$

$$\frac{d^2r}{d\tau^2} = -\frac{M}{r^2}\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + (r-2GM)\left(\frac{d\phi}{d\tau}\right)^2 \tag{46}$$

$$\frac{d^2\phi}{d\tau^2} = -\frac{2}{r}\frac{dr}{d\tau}\frac{d\phi}{d\tau}. \tag{47}$$

$$\tag{48}$$

2c

We can now compare this to the geodesic equations written in terms of the Christoffel symbols

$$\frac{d^2x^\mu}{d\tau^2} = -\Gamma^\mu_{\kappa\lambda}\frac{dx^\kappa}{d\tau}\frac{dx^\lambda}{d\tau} \tag{49}$$

to find

$$\Gamma^t_{tr} = \frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-1} \tag{50}$$

$$\Gamma^r_{tt} = \frac{M}{r^2}\left(1-\frac{2M}{r}\right) \quad \Gamma^r_{rr} = -\frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-1} \quad \Gamma^r_{\phi\phi} = -\left(r-2GM\right) \tag{51}$$

$$\Gamma^\phi_{r\phi} = \frac{1}{r} \tag{52}$$

Careful that when the two lowered indices are different, the Christoffel symbols are symmetric in these indices and they occur twice in the sum so that we get a factor of two taken off the corresponding coefficient.

3 *(Hartle 22.16,22.17) A four-dimensional spacetime has the line element*

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2+\sin^2\theta d\phi^2\right) \tag{53}$$

(a) Find the Christoffel symbols for this metric by whatever method.
(b) Show that the Ricci curvature of this metric vanishes.

3a

It would seem to be very foolish not to use all the derivatives we calculated in 2/ so we will use the geodesics rather than the derivative formulae to find the Christoffel symbols. You don't have to repeat much of the calculation since we have just added a variable from 2/. I am just going to cut and paste, adding in the new angular dependence.

Now we have

$$L'(x^\mu, dx^\mu/d\tau) = \left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\theta}{d\tau}\right)^2 - r^2\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2 \tag{54}$$

There is nothing for it but to calculate the derivatives. I find

$$\frac{\partial L'}{\partial t} = 0 \tag{55}$$

$$\frac{\partial L'}{\partial r} = \frac{2M}{r^2}\left(\frac{dt}{d\tau}\right)^2 + \frac{2M}{r^2}\left(1-\frac{2M}{r}\right)^{-2}\left(\frac{dr}{d\tau}\right)^2 - 2r\left(\frac{d\theta}{d\tau}\right)^2 - 2r\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2 \tag{56}$$

$$\frac{\partial L'}{\partial \theta} = -2r^2\sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2 \tag{57}$$

$$\frac{\partial L'}{\partial \phi} = 0 \tag{58}$$

$$\tag{59}$$

and

$$\frac{\partial L'}{\partial(dt/d\tau)} = 2\left(1-\frac{2M}{r}\right)\frac{dt}{d\tau} \tag{60}$$

$$\frac{\partial L'}{\partial(dr/d\tau)} = -2\left(1-\frac{2M}{r}\right)^{-1}\frac{dr}{d\tau} \tag{61}$$

$$\frac{\partial L'}{\partial(d\theta/d\tau)} = -2r^2\frac{d\theta}{d\tau} \tag{62}$$

$$\frac{\partial L'}{\partial(d\phi/d\tau)} = -2r^2\sin^2\theta\frac{d\phi}{d\tau} \tag{63}$$

When we evaluate $d/d\tau$ we need to use the product and the chain rule, we need to do partial derivatives with respect to all of the variables and their derivatives. We find

$$
\begin{aligned}
\frac{d}{d\tau}\frac{\partial L'}{\partial(dt/d\tau)} &= 2\left(1-\frac{2M}{r}\right)\frac{d^2t}{d\tau^2}+\frac{4M}{r^2}\frac{dt}{d\tau}\frac{dr}{d\tau} && (64)\\
\frac{d}{d\tau}\frac{\partial L'}{\partial(dr/d\tau)} &= -2\left(1-\frac{2M}{r}\right)^{-1}\frac{d^2r}{d\tau^2}+\frac{4M}{r^2}\left(1-\frac{2M}{r}\right)^{-2}\left(\frac{dr}{d\tau}\right)^2 && (65)\\
\frac{d}{d\tau}\frac{\partial L'}{\partial(d\theta/d\tau)} &= -2r^2\frac{d^2\theta}{d\tau^2}-4r\frac{dr}{d\tau}\frac{d\theta}{d\tau} && (66)\\
\frac{d}{d\tau}\frac{\partial L'}{\partial(d\phi/d\tau)} &= -2r^2\sin^2\theta\frac{d^2\phi}{d\tau^2}-4r\sin^2\theta\frac{dr}{d\tau}\frac{d\phi}{d\tau}-4r^2\sin\theta\cos\theta\frac{d\theta}{d\tau}\frac{d\phi}{d\tau}. && (67)
\end{aligned}
$$

We can now rewrite the Euler-Lagrange equations to give us the accelerations

$$
\begin{aligned}
\frac{d^2t}{d\tau^2} &= -\frac{2M}{r^2}\left(1-\frac{2M}{r}\right)^{-1}\frac{dt}{d\tau}\frac{dr}{d\tau} && (68)\\
\frac{d^2r}{d\tau^2} &= -\frac{M}{r^2}\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2+\frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2+(r-2GM)\left(\frac{d\theta}{d\tau}\right)^2+(r-2GM)\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2 && (69)\\
\frac{d^2\theta}{d\tau^2} &= -\frac{2}{r}\frac{dr}{d\tau}\frac{d\phi}{d\tau}+\sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2 && (70)\\
\frac{d^2\phi}{d\tau^2} &= -\frac{2}{r}\frac{dr}{d\tau}\frac{d\phi}{d\tau}-2\cot\theta\frac{d\theta}{d\tau}\frac{d\phi}{d\tau}. && (71)
\end{aligned}
$$

We can now compare this to the geodesic equations written in terms of the Christoffel symbols

$$\frac{d^2x^\mu}{d\tau^2}=-\Gamma^\mu_{\kappa\lambda}\frac{dx^\kappa}{d\tau}\frac{dx^\lambda}{d\tau} \tag{72}$$

to find

$$
\begin{aligned}
\Gamma^t_{tr} &= \frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-1} && (73)\\
\Gamma^r_{tt} &= \frac{M}{r^2}\left(1-\frac{2M}{r}\right) \quad \Gamma^r_{rr}=-\frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-1} \quad \Gamma^r_{\theta\theta}=-(r-2M) \quad \Gamma^r_{\phi\phi}=-(r-2M)\sin^2\theta && (74)\\
\Gamma^\theta_{r\theta} &= \frac{1}{r} \quad \Gamma^\theta_{\phi\phi}=-\sin\theta\cos\theta && (75)\\
\Gamma^\phi_{r\phi} &= \frac{1}{r} \quad \Gamma^\phi_{\theta\phi}=\cot\theta && (76)
\end{aligned}
$$

Hopefully everyone had the gumption to check look at the answer in the back of the text books before attempting this.

3b

Now we are supposed to find the Ricci tensor, which amounts to checking that the Schwarzschild geometry is a solution to Einstein's equations.

We need the definition of the Riemann curvature tensor elements.

$$R^\rho_{\sigma\mu\nu}=\partial_\mu\Gamma^\rho_{\nu\sigma}-\partial_\nu\Gamma^\rho_{\mu\sigma}+\Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma}-\Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \tag{77}$$

You might recall that the form of the curvature tensor with all indices lowered is antisymmetric in the first pair of indices as well as the second pair. Since the metric is diagonal here we can conclude that $R^\rho_{\sigma\mu\nu}=0$ whenever $\rho=\sigma$ or $\mu=\nu$.

In order to find the Ricci tensor we only need to look at terms where $\rho=\mu$ since the Ricci tensor is defined as a trace over the curvature tensor. So we look for terms like $R^\rho_{\sigma\rho\nu}$. Also recall that the Ricci tensor is symmetric so we only need to contemplate each possible pair (σ,ν) once.

Lets look at

$$\begin{aligned}
R^t_{rtr} &= \partial_t \Gamma^t_{rr} - \partial_r \Gamma^t_{tr} + \Gamma^t_{t\lambda}\Gamma^\lambda_{rr} - \Gamma^t_{r\lambda}\Gamma^\lambda_{tr} && (78)\\
&= 0 - \partial_r \left[\frac{M}{r^2}\left(1 - \frac{2M}{r}\right)^{-1}\right] - 2\left(\frac{M}{r^2}\right)^2 \left(1-\frac{2M}{r}\right)^{-2} && (79)\\
&= \frac{2M}{r^3}\left(1-\frac{2M}{r}\right)^{-1} && (80)
\end{aligned}$$

Here I've made use of the fact that λ is forced to be a single value in the last two terms since there is only one non-zero Christoffel symbol with a raised t index. Then I've also noticed that $\Gamma^t_{tr} = -\Gamma^r_{rr}$.

Now we could consider $R^t_{r\mu\nu}$ for $(\mu,\nu) \neq (t,r)$ and $\mu \neq \nu$ that is $(t,\theta),(t,\phi),(r,\theta),(r,\phi),(\theta,\phi)$. It is possible to see right away that you need either μ or ν to be equal to t or r or else all terms are zero since Γ^t_{tr} is the only non-zero Christoffel symbol with a raised t index. This rules out (θ,ϕ). Going through each of these possibilities one finds that all these combinations of indices result in situations where every term in the defintion is zero by inspection. Moreover we only need to consider $(t,\theta),(t,\phi)$ since these are the only two that could contibute to the curvature tensor.

Next we could consider the entries $R^t_{\theta t\theta}, R^t_{\theta t\phi}, R^t_{\phi t\phi}$.

$$\begin{aligned}
R^t_{\theta t\theta} &= \partial_t \Gamma^t_{\theta\theta} - \partial_\theta \Gamma^t_{t\theta} + \Gamma^t_{t\lambda}\Gamma^\lambda_{\theta\theta} - \Gamma^t_{\theta\lambda}\Gamma^\lambda_{t\theta} && (81)\\
&= 0 - 0 + \Gamma^t_{tr}\Gamma^r_{\theta\theta} - 0 && (82)\\
&= -\frac{M}{r} && (83)
\end{aligned}$$

By inspection we can see $R^t_{\theta t\phi} = 0$ while

$$\begin{aligned}
R^t_{\phi t\phi} &= \Gamma^t_{tr}\Gamma^r_{\phi\phi} && (84)\\
&= -\frac{M}{r}\sin^2\theta && (85)
\end{aligned}$$

So we can move on to values with a raised r index. Starting with

$$R^r_{tr\nu} = g^{rr}R_{rtr\nu} = -g^{rr}R_{trr\nu} = -g^{rr}g_{tt}R^t_{rr\nu} \tag{86}$$

From this we can use our results above to find that the only value of ν that will give us anything non-zero is $\nu = t$ so we find $R^r_{trt} = g^{rr}g_{tt}R^t_{rtr}$ and so

$$R^r_{trt} = -\frac{2M}{r^3}\left(1 - \frac{2M}{r}\right). \tag{87}$$

Next we can consider values of the form $R^r_{\theta r\nu}$. We can use the fact that $R^t_{rr\theta} = 0$ and similar index shuffling to conclude that $\nu \neq t$.

So we can look at

$$\begin{aligned}
R^r_{\theta r\theta} &= \partial_r \Gamma^r_{\theta\theta} - \partial_\theta \Gamma^r_{r\theta} + \Gamma^r_{r\lambda}\Gamma^\lambda_{\theta\theta} - \Gamma^r_{\theta\lambda}\Gamma^\lambda_{r\theta} && (88)\\
&= \partial_r \Gamma^r_{\theta\theta} + \Gamma^r_{rr}\Gamma^r_{\theta\theta} - \Gamma^r_{\theta\theta}\Gamma^\theta_{r\theta} && (89)\\
&= -\frac{M}{r} && (90)
\end{aligned}$$

but we can inspect $R^r_{\theta r\phi}$ to see that it is zero.

Moving on to $R^r_{\phi r\nu}$ we can rule out $\nu = t, r$ by shuffling around the indices on our previous results and so we are left with

$$\begin{aligned}
R^r_{\phi r\phi} &= \partial_r \Gamma^r_{\phi\phi} - \partial_\phi \Gamma^r_{r\phi} + \Gamma^r_{r\lambda}\Gamma^\lambda_{\phi\phi} - \Gamma^r_{\phi\lambda}\Gamma^\lambda_{r\phi} && (91)\\
&= \partial_r \Gamma^r_{\phi\phi} + \Gamma^r_{rr}\Gamma^r_{\phi\phi} - \Gamma^r_{\phi\phi}\Gamma^\phi_{r\phi} && (92)\\
&= -\frac{M}{r}\sin^2\theta && (93)
\end{aligned}$$

Now we can consider entries of the form $R^{\theta}_{\phi\theta\nu}$. It's easy to see by inspecting the terms that $R^{\theta}_{\phi\theta t} = 0 = R^{\theta}_{\phi\theta r}$ so we are left with

$$\begin{aligned} R^{\theta}_{\phi\theta\phi} &= \partial_\theta \Gamma^{\theta}_{\phi\phi} - \partial_\phi \Gamma^{\theta}_{\theta\phi} + \Gamma^{\theta}_{\theta\lambda}\Gamma^{\lambda}_{\phi\phi} - \Gamma^{\theta}_{\phi\lambda}\Gamma^{\lambda}_{\theta\phi} && (94) \\ &= \partial_\theta \Gamma^{\theta}_{\phi\phi} + \Gamma^{\theta}_{\theta r}\Gamma^{r}_{\phi\phi} - \Gamma^{\theta}_{\phi\phi}\Gamma^{\phi}_{\theta\phi} && (95) \\ &= \frac{2M}{r}\sin^2\theta && (96) \end{aligned}$$

Now in fact we can check by looking it up that these are in fact all the non-zero elements of the curvature tensor up to reshuffling the indices, this despite there being some possible pairs that we haven't checked yet. We have however checked all the ones that can possibly contribute to the Ricci scalar.

Now since we haven't found any terms like $R^{\rho}_{\sigma\rho\nu}$ for $\sigma \neq \nu$ we know that all the off-diagonal elements of the Ricci tensor are zero.

Lets look at the diagonal elements one by one.

$$\begin{aligned} R_{tt} &= R^{r}_{trt} + R^{\theta}_{t\theta t} + R^{\phi}_{t\phi t} && (97) \\ &= -\frac{2M}{r^3}\left(1 - \frac{2M}{r}\right) + g_{tt}g^{\theta\theta}R^{t}_{\theta t\theta} + g_{tt}g^{\phi\phi}R^{t}_{\phi t\phi} && (98) \\ &= -\frac{2M}{r^3}\left(1 - \frac{2M}{r}\right) - \left(1 - \frac{2M}{r}\right)\frac{1}{r^2}\left(-\frac{M}{r}\right) - \left(1 - \frac{2M}{r}\right)\frac{1}{r^2\sin^2\theta}\left(-\frac{M}{r}\sin^2\theta\right) = 0 && (99) \end{aligned}$$

So we are off to a good start.

$$\begin{aligned} R_{rr} &= R^{t}_{rtr} + R^{\theta}_{r\theta r} + R^{\phi}_{r\phi r} && (100) \\ &= \frac{2M}{r^3}\left(1 - \frac{2M}{r}\right)^{-1} + g_{rr}g^{\theta\theta}R^{r}_{\theta r\theta} + g_{rr}g^{\phi\phi}R^{r}_{\phi r\phi} && (101) \\ &= \frac{2M}{r^3}\left(1 - \frac{2M}{r}\right)^{-1} + \left(1 - \frac{2M}{r}\right)^{-1}\frac{1}{r^2}\left(-\frac{M}{r}\right) + \left(1 - \frac{2M}{r}\right)^{-1}\frac{1}{r^2\sin^2\theta}\left(-\frac{M}{r}\sin^2\theta\right) = 0 && (102) \end{aligned}$$

Again we get zero.

$$\begin{aligned} R_{\theta\theta} &= R^{t}_{\theta t\theta} + R^{r}_{\theta r\theta} + R^{\phi}_{\theta\phi\theta} && (103) \\ &= -\frac{M}{r} - \frac{M}{r} + g^{\phi\phi}g_{\theta\theta}R^{\theta}_{\phi\theta\phi} && (104) \\ &= -\frac{2M}{r} + r^2\frac{1}{r^2\sin^2\theta}\frac{2M}{r}\sin^2\theta = 0. && (105) \end{aligned}$$

Finally

$$\begin{aligned} R_{\phi\phi} &= R^{t}_{\phi t\phi} + R^{r}_{\phi r\phi} + R^{\theta}_{\phi\theta\phi} && (106) \\ &= -\frac{M}{r}\sin^2\theta - \frac{M}{r}\sin^2\theta + \frac{2M}{r}\sin^2\theta = 0. && (107) \end{aligned}$$

So we have shown explicitly that the Schwarzschild metric satisfies Einstein's equations for empty space.