Ex 1 $\Gamma_{\mu\lambda\nu} + \Gamma_{\nu\lambda\mu} = g_{\mu\nu,\lambda}$

Proof: LHS: $= \frac{1}{2}(\cancel{g_{\lambda\mu,\nu}} + g_{\nu\mu,\lambda} - \cancel{g_{\mu\lambda,\nu}}) + \frac{1}{2}(\cancel{g_{\lambda\mu,\nu}} + g_{\mu\nu,\lambda} - \cancel{g_{\nu\lambda,\mu}})$.

$= g_{\mu\nu,\lambda}$.

Ex 2 $g^{\mu\nu}{}_{,\lambda} = -g^{\mu\sigma}\Gamma^{\nu}_{\lambda\sigma} - g^{\nu\sigma}\Gamma^{\mu}_{\lambda\sigma}$.

$\boxed{19/08/08}$ Proof: $\frac{\partial}{\partial x^\lambda}(g^{\mu\sigma}g_{\sigma\nu}) = \frac{\partial}{\partial x^\lambda}(\delta^\mu_\nu) = 0$

$\Rightarrow 0 = g^{\mu\sigma}\frac{\partial}{\partial x^\lambda}g_{\sigma\nu} + \frac{\partial}{\partial x^\lambda}(g^{\mu\sigma})g_{\sigma\nu}$

ie. $g^{\mu\sigma}g_{\sigma\nu,\lambda} = -g^{\mu\sigma}{}_{,\lambda}g_{\sigma\nu}$

Multiply by $g^{\nu\rho}$ $\Rightarrow g^{\nu\rho}g^{\mu\sigma}g_{\sigma\nu,\lambda} = -\underbrace{g^{\nu\rho}g_{\sigma\nu}}_{\delta^\rho_\sigma}g^{\mu\sigma}{}_{,\lambda}$

$\Rightarrow g^{\mu\rho}{}_{,\lambda} = -g^{\nu\rho}g^{\mu\sigma}g_{\sigma\nu,\lambda}$

$= -g^{\nu\rho}g^{\mu\sigma}(\Gamma_{\sigma\lambda\nu} + \Gamma_{\nu\lambda\sigma})$

$= -g^{\nu\rho}\Gamma^{\mu}_{\lambda\nu} - g^{\mu\sigma}\Gamma^{\rho}_{\lambda\sigma}$

- Absolute Derivatives

A related concept is that of absolute derivative of a tensor along a curve $C: t_0 \le t \le t_1$

A related concept is that of absolute derivative of a tensor defined along C (e.g. tangent vector).

E.g. $u^\mu(t)$ – contravariant vector defined along C (e.g. tangent vector).

However, $\frac{du^\mu(t)}{dt}$ is NOT a vector.

So define absolute derivative:

$$\left.\begin{aligned}\frac{Du^\lambda}{Dt} &\equiv \frac{du^\lambda}{dt} + \Gamma^\lambda_{\mu\nu}u^\mu\frac{dx^\nu}{dt}\\ \frac{Du_\lambda}{Dt} &\equiv \frac{du_\lambda}{dt} - \Gamma^{\mu}_{\lambda\nu}u_\mu\frac{dx^\nu}{dt}\end{aligned}\right\}\text{ Both are vectors!}$$

- Gradients, curls and divergences.

1. Given a scalar field $\Phi(x)$, define covariant vector

$$(\text{grad}\,\Phi)_i = \Phi_{,i}$$

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2. Given a vector field V^i

$\Rightarrow V = V^i e_i$

Define

$$\operatorname{div} V = V^i{}_{;i} = V^i{}_{,i} + \Gamma^i_{ki} V^k$$

Now can be shown

$$\Gamma^i_{ki} = \frac{\partial}{\partial x^k}\left(\ln\sqrt{g(x)}\right) \qquad (*)$$

$$g(x) = \det\left(g_{ij}(x)\right)$$

<u>Proof</u>

$$\Gamma^i_{ki} = \frac{1}{2} g^{ij}\left(g_{jk,i} + g_{ij,k} - g_{ki,j}\right)$$

$$= \frac{1}{2} g^{ij} \frac{\partial}{\partial x^k} g_{ji} + \frac{1}{2}\left(\underbrace{g^{ij} g_{jk,i}}_{g^{ji} g_{ik,j} = g^{ij} \cancel{g_{ki,j}}} - g^{ij} \cancel{g_{ki,j}}\right)$$

$$= \frac{1}{2} g^{il} \frac{\partial}{\partial x^k} g_{li}$$

By formula for any matrix

$$\mathrm{Tr}\left(m^{-1}(x) \frac{\partial}{\partial x^\lambda} m(x)\right) = \frac{\partial}{\partial x^\lambda} \ln \det M(x)$$

<u>Proof:</u>

$$\delta(\ln \det M(x)) = \ln\det(M + \delta m) - \ln\det M$$

$$= \ln\left(\frac{\det(m + \delta m)}{\det m}\right) = \ln \det\left(m^{-1}(m + \delta m)\right)$$

$$= \ln\det(1 + m^{-1}\delta m) =$$

$$\simeq \ln(1 + \mathrm{Tr}\, m^{-1}\delta m) \quad \text{(for } \delta m \text{ very small)}.$$

$$\cong \mathrm{Tr}(m^{-1}\delta m)$$

Divide both sides by δx^λ

$$\Rightarrow \frac{\partial}{\partial x^\lambda} \ln\det M = \mathrm{Tr}\left(m^{-1} \frac{\partial}{\partial x^\lambda} m\right).$$

$$\Rightarrow \Gamma^i_{ki} = \frac{1}{2} \frac{\partial}{\partial x^k} \ln g(x)$$

$$= \frac{\partial}{\partial x^k} \ln\sqrt{g(x)} \quad \left(= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k}\sqrt{g}\right)$$

$$\Rightarrow \underline{\underline{\operatorname{div} V}} = V^i{}_{,i} + (\ln\sqrt{g})_{,k} V^k$$

$$= V^i{}_{,i} + \frac{1}{\sqrt{g}}(\sqrt{g})_{,k} V^k = \frac{1}{\sqrt{g}}\left(\sqrt{g}\, V^k\right)_{,k}$$

So divergence of a vector is:

$$\operatorname{div} V = \frac{1}{\sqrt{g}}\left(\sqrt{g}\, V^k\right)_{,k}$$

3. Essentially, no extension of curl to >3D.

4. Laplacian of Φ.

$$\nabla^2\Phi = \operatorname{div}(g^{kl}\Phi_{,l}e_k)$$

$$= \frac{1}{\sqrt{g}}(\sqrt{g}\, g^{kl}\Phi_{,l})_{,k}$$

Another proof of the Laplacian.

$$\nabla^2\Phi = \frac{1}{\sqrt{g}}(\sqrt{g}\, g^{kl}\Phi_{,l})_{,k}$$

$$\nabla^2\Phi = g^{kl}\Phi_{;k;l} = g^{kl}\Phi_{,k;l}$$

$$= g^{kl}(\Phi_{,k,l} - \Gamma^{i}_{kl}\Phi_{,i})$$

$$= g^{kl}\Phi_{,k,l} - g^{ki}\Gamma_{ki}{}^{l}\Phi_{,l}$$

$$\text{2nd term} = -g^{ki}\Gamma_{ki}^{l}\Phi_{,l} - g^{lk}\Gamma_{ik}^{i}\Phi_{,l} + g^{lk}\Gamma_{ik}^{i}\Phi_{,l}$$

$$= -(g^{ki}\Gamma_{ki}^{l} + g^{lk}\Gamma_{ik}^{i})\Phi_{,l} + g^{lk}\Gamma_{ik}^{i}\Phi_{,l}$$

$$= g^{kl}{}_{,k}\Phi_{,l} + g^{kl}\left(\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}\sqrt{g}\right)\Phi_{,l}$$

$$\Rightarrow \nabla^2\Phi = \frac{1}{\sqrt{g}}(\sqrt{g}\, g^{kl}\Phi_{,l})_{,k}$$

- <u>Geodesics</u>

Geodesic is the analogue in curved space or manifold of straight line in E_n. Let's write

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

$ds^2 > 0$, < 0 or $= 0$ in pseudo-Riemann space, if $ds^2 \neq 0$ write

$$ds^2 = \varepsilon g_{\mu\nu}\, dx^\mu dx^\nu, \qquad \varepsilon = \begin{cases} +1 & \text{if } ds^2 > 0 \\ -1 & \text{if } ds^2 < 0 \end{cases}$$

$$\Rightarrow ds = \sqrt{\varepsilon g_{\mu\nu} dx^\mu dx^\nu} \quad \text{— distance between neighbouring points.}$$

Suppose we have a curve C: $x^\mu = x^\mu(t)$, t-parameter, $t_1 \le t \le t_2$.

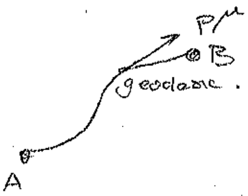
On C: $dx^\mu = \dfrac{dx^\mu}{dt} dt \equiv p^\mu dt$ ← tangent vector

$$\Rightarrow ds = \sqrt{\varepsilon g_{\mu\nu} p^\mu p^\nu}\, dt.$$

(Assume ε is a constant on C).

<u>Length of curve</u>: $S_{AB} = \displaystyle\int_A^B ds = \int_{t_1}^{t_2} \sqrt{\varepsilon g_{\mu\nu} p^\mu p^\nu}\, dt$

A geodesic joining points A and B is a curve whose length has stationary values w.r.t. arb. small variation of curve, end points being held fixed;



$$\delta \int_A^B ds = 0$$

$\delta x_A = 0 = \delta x_B$.

Have $S_{AB} = \displaystyle\int_{t_1}^{t_2} L(x, p)\, dt$, where $L(x,p) = \sqrt{\varepsilon g_{\mu\nu} p^\mu p^\nu}$ ← Lagrangian.

$$\Rightarrow \delta S_{AB} = 0 \Rightarrow 0 = \int_{t_1}^{t_2} \frac{\partial L}{\partial x^\mu}\delta x^\mu + \frac{\partial L}{\partial p^\mu}\delta p^\mu \Big) dt$$

$$= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x^\mu}\delta x^\mu + \frac{\partial L}{\partial p^\mu}\frac{d}{dt}(\delta x^\mu)\right) dt$$

$p^\mu = \dfrac{dx^\mu}{dt}$

$\delta p^\mu = \dfrac{d\delta x^\mu}{dt}$

$$= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x^\mu} - \frac{d}{dt}\left(\frac{\partial L}{\partial p^\mu}\right)\right)\delta x^\mu dt + \frac{\partial L}{\partial p^\mu}\delta x^\mu \Big|_{t_1}^{t_2}$$

$(\because \delta x^\mu(t_2) = 0,\ \delta x^\mu(t_1) = 0)$

$\therefore \delta x^\mu$ is arb. variation

$$\Rightarrow \frac{\partial L}{\partial x^\mu} - \frac{d}{dt}\left(\frac{\partial L}{\partial p^\mu}\right) = 0 \quad \text{— Euler's eqn}$$

$$\text{or} \quad \frac{\partial L}{\partial x^\mu} - \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) = 0$$

Set $L = \sqrt{\varepsilon W}$

$$\Rightarrow \frac{\partial L}{\partial x^\mu} = \varepsilon \frac{\partial W}{\partial x^\mu} \cdot \frac{1}{2\sqrt{\varepsilon W}}$$

$$\frac{\partial L}{\partial p^\mu} = \varepsilon \frac{\partial W}{\partial p^\mu} \cdot \frac{1}{2\sqrt{\varepsilon W}}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial p^\mu}\right) = \varepsilon \frac{d}{dt}\left(\frac{\partial W}{\partial p^\mu}\right) \cdot \frac{1}{2\sqrt{\varepsilon W}} + \frac{\partial W}{\partial p^\mu}\left(-\frac{1}{4\sqrt{\varepsilon W}} \frac{dW}{dt}\right)$$

$$\Rightarrow \frac{\partial W}{\partial x^\mu} - \frac{d}{dt}\left(\frac{\partial W}{\partial p^\mu}\right) + \frac{1}{2W} \frac{\partial W}{\partial p^\mu} \frac{dW}{dt} = 0$$

So far, t is arb. parameter. To simplify now we choose $t = s$, arclength along the geodesic,

$$p^\mu = \frac{dx^\mu}{ds}, \quad W = g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = \varepsilon$$

$\Rightarrow \frac{dW}{dt} = 0$ in this case

$$\Rightarrow \frac{\partial W}{\partial x^\mu} - \frac{d}{ds}\left(\frac{\partial W}{\partial p^\mu}\right) = 0$$

$$W = g_{\mu\nu}(x) \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = g_{\mu\nu}(x) p^\mu p^\nu,$$

$$\Rightarrow \frac{\partial W}{\partial p^\mu} = \frac{\partial}{\partial p^\mu}\left(g_{\lambda\nu} p^\lambda p^\nu\right)$$

$$= g_{\lambda\nu} \delta^\lambda_\mu p^\nu + g_{\lambda\nu} p^\lambda \delta^\nu_\mu$$

$$= 2 g_{\mu\nu} p^\nu$$

$$\frac{\partial W}{\partial x^\mu} = g_{\lambda\nu,\mu} p^\lambda p^\nu$$

geodesic eqn.

$$\text{So} \quad g_{\lambda\nu,\mu} p^\lambda p^\nu - \frac{d}{ds}\left(2 g_{\mu\nu} p^\nu\right) = 0.$$

$$\text{Now} \quad \frac{d}{ds}\left(2 g_{\mu\nu} p^\nu\right) = \frac{\partial g_{\mu\nu}}{\partial x^\lambda} \frac{dx^\lambda}{ds} p^\nu + g_{\mu\nu} \frac{dp^\nu}{ds}$$

$$= \underbrace{g_{\mu\nu,\lambda} p^\lambda p^\nu}_{= \frac{1}{2}(g_{\mu\nu,\lambda} + g_{\mu\lambda,\nu}) p^\lambda p^\nu} + g_{\mu\nu} \frac{dp^\nu}{ds}$$

$$\Rightarrow g_{\mu\nu} \frac{dp^\nu}{ds} + \underbrace{\frac{1}{2}\left(g_{\lambda\mu,\nu} + g_{\mu\nu,\lambda} - g_{\lambda\nu,\mu}\right)}_{\Gamma_{\lambda\nu\mu} (= \Gamma_{\nu\lambda\mu})} p^\lambda p^\nu = 0$$

$$\Rightarrow g_{\mu\nu} \frac{dp^\nu}{ds} + \Gamma_{\lambda\nu\mu} p^\lambda p^\nu = 0$$

Multiply by $g^{\sigma\mu}$ on both side

$$\Rightarrow \frac{dp^\sigma}{ds} + \Gamma^\sigma_{\lambda\nu} p^\lambda p^\nu = 0.$$

or $$\boxed{\frac{d^2 x^\sigma}{ds^2} + \Gamma^\sigma_{\lambda\nu} \frac{dx^\lambda}{ds} \frac{dx^\nu}{ds} = 0}$$ Equation of geodesics

- Eqn of geodesic is covariant $\Rightarrow$ i.e. have same form in all coord systems under $x \to x'$

$$\frac{d^2 x'^\sigma}{ds^2} + \Gamma'^{\,\sigma}_{\lambda\nu} \frac{dx'^\lambda}{ds} \frac{dx'^\nu}{ds} = 0$$

- To fix a soln, we must be given initial values of x^μ, $\frac{dx^\mu}{ds}$ and boundary values x^μ_A, x^μ_B

- In Euclidean space E_n: geodesic eqn is $\Rightarrow \frac{d^2 x^i}{ds^2} = 0$

$$\Rightarrow \frac{dx^i}{ds} = \text{const} = \dot{x}^i(0)$$

$$\Rightarrow x^i(s) = \dot{x}^i(0)s + x^i(0).$$

$\longrightarrow$ straight line

$\boxed{22/08/08}$

Ex: In E_2, using polar coords $(x^1, x^2) = (r, \phi)$

$$ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\phi$$

$$\Rightarrow g_{11} = 1, \; g_{22} = r^2, \; g_{12} = g_{21} = 0$$

Inverse of (g_{ij}): $g^{11} = 1$, $g^{22} = \frac{1}{r^2}$, $g^{12} = g^{21} = 0$

$$\Rightarrow \Gamma_{122} \; (= \Gamma_{212}) = r, \; \Gamma_{221} = -r, \; \text{others} = 0.$$

$$\Gamma_{12}^{\;2} \; (= \Gamma_{21}^{\;2}) = \frac{1}{r}, \; \Gamma_{22}^{\;1} = -r, \; \text{others} = 0$$

$\Rightarrow$ eqns of geodesics $(r(s), \phi(s))$ are

$$\frac{d^2 r}{ds^2} + \Gamma_{22}^{\;1} \left(\frac{d\phi}{ds}\right)^2 = 0$$

$$\frac{d^2 \phi}{ds^2} + \Gamma_{12}^{\;2} \frac{dr}{ds} \frac{d\phi}{ds} + \Gamma_{21}^{\;2} \frac{d\phi}{ds} \frac{dr}{ds} = 0$$

$$\Rightarrow \begin{cases} \dfrac{d^2 r}{ds^2} - r\left(\dfrac{d\phi}{ds}\right)^2 = 0 \\[2ex] \dfrac{d^2\phi}{ds^2} + \dfrac{2}{r}\dfrac{dr}{ds}\dfrac{d\phi}{ds} = 0 \end{cases}$$

Good exercise: put

$$r(s) = \sqrt{x(s)^2 + y(s)^2}$$

$$\phi(s) = \tan^{-1}\left(\frac{y(s)}{x(s)}\right)$$

show that the geodesic eq^n reduces to

$$\frac{d^2x}{ds^2} = 0 = \frac{d^2y}{ds^2}$$

- Consider DE's $\frac{\partial W}{\partial x^\mu} = \frac{d}{dt}\left(\frac{\partial W}{\partial p^\mu}\right)$ (*)

Multiply on both sides by $p^\mu = \frac{dx^\mu}{dt}$

$$\Rightarrow \frac{\partial w}{\partial x^\mu}\frac{dx^\mu}{dt} = p^\mu \frac{d}{dt}\left(\frac{\partial w}{\partial p^\mu}\right)$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x^\mu}\frac{dx^\mu}{dt} + \frac{\partial w}{\partial p^\mu}\frac{dp^\mu}{dt}$$

$$= p^\mu \frac{d}{dt}\left(\frac{\partial w}{\partial p^\mu}\right) + \frac{dp^\mu}{dt}\frac{\partial w}{\partial p^\mu}$$

use above eqn

$$= \frac{d}{dt}\left(p^\mu \frac{\partial w}{\partial p^\mu}\right)$$

$$\Rightarrow \frac{d}{dt}\left(p^\mu \frac{\partial w}{\partial p^\mu} - w\right) = 0$$

or $p^\mu \frac{\partial w}{\partial p^\mu} - W = \text{const} \Rightarrow$ first integral of the system.

Now if we put: $w = g_{\mu\nu} p^\mu p^\nu$ and differentiate w.r.t p^μ, we get

$$p^\mu \frac{\partial w}{\partial p^\mu} = 2w$$

$\Rightarrow p^\mu \frac{\partial w}{\partial p^\mu} - W = W \Rightarrow$ have $W = \text{const}$

Put $t = s$ in (*), have eq^n of geodesic and

$$W = g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = \text{const}$$

$\Rightarrow$ it is consistent to suppose ε is const along geodesic.

Suppose start curve in <u>null direction</u> (i.e. $\frac{dx^\mu}{dt} = 0$) and so $W = 0$ and assume (*) holds for some parameter t, with

$$W = g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}$$

$\Rightarrow$ get $W = \text{const} = 0$ along the curve and geodesic

equation $\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\lambda\nu}\frac{dx^\lambda}{dt}\frac{dx^\nu}{dt} = 0$

or $\frac{D}{Dt}\left(\frac{dx^\mu}{dt}\right) = 0$ call this curve the <u>null geodesic</u>.

Solution again determined by giving $x^\mu(0)$, $\frac{dx^\mu(0)}{dt}$ but tangent vector must be a <u>null vector</u>, i.e. $(w=) g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = 0$.

If vector a^μ along C is <u>also</u> vector field

$$\Rightarrow \frac{Da^\mu}{Dt} = \frac{da^\mu}{dt} + \Gamma^\mu_{\lambda\nu} a^\lambda \frac{dx^\nu}{dt}$$

$$= \frac{\partial a^\mu}{\partial x^\nu}\frac{dx^\nu}{dt} + \Gamma^\mu_{\lambda\nu} a^\lambda \frac{dx^\nu}{dt}$$

$\because a^\mu$ is also a vector field

$$= a^\mu{}_{;\nu}\frac{dx^\nu}{dt}$$

Similarly if $a_\mu(x)$ is covariant vector field, then along C:

$$\frac{Da_\mu}{Dt} = a_{\mu;\nu}\frac{dx^\nu}{dt}$$

If $\frac{Da^\mu}{Dt} = 0$ along C (note $\frac{Da^\mu}{Dt}$ is a covariant eqn, i.e. has same form in all coords), say a^μ is propagated parallel along C.

For such a vector $\frac{da^\mu}{dt} = -\Gamma^\mu_{\lambda\nu}\frac{dx^\nu}{dt}a^\lambda$

Given $a^\mu(t_0)$ at some initial t_0, above equation defines vector $a^\mu(t)$ for all t.

$\Rightarrow$ Any vector (or generalising: tensor) can be defined along a curve by means of parallel propagation.