MATH4105 General Relativity Problem Sheet 6

1 (Based on Hartle including problems 7.11,7.12) Consider a metric sometimes known as the Alcubierre warp drive. There is one of these metrics for any curve in space of the form $x_s(t), y = 0, z = 0$. We can define the velocity associated with the curve $v_s(t) = dx_s/dt$ and the distance of any spatial point from the curve $r_s(t) = (x - x_s(t))^2 + y^2 + z^2$. Finally we define a smooth positive function $f(r_s)$ such that $f(0) = 1$ and $f(r_s)$ vanishes when $r_s > R$ for some R. The line element specifying the metric is then

$$ds^2 = -dt^2 + (dx - v_s f(r_s)dt)^2 + dy^2 + dz^2. \tag{1}$$

We will describe $x_s(t)$ as the path of a space ship undergoing a "warp-drive"

(a) Find the paths of light beams in this metric and show that at every point along $x_s(t)$ the four-velocity of the ship lies inside the forward light cone.

(b) How much ship time elapses on a trip between stations that takes co-ordinate time T?

(c) Show that the path $x^\mu(t) = (t, x_s(t), 0, 0)$ is a geodesic of this metric.

2 (Hartle 8.3) A three-dimensional spacetime has the line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right) dr^2 + r^2 d\phi^2 \tag{2}$$

(a) Find the explicit Lagrangian for the variational principle for geodesics in this spacetime in these co-ordinates.

(b) Using the results of (a) write out the components of the geodesic equation by computing them from the Lagrangian.

(c) Read off the non-zero Christoffel symbols for this metric from your results in (b).

3 (Hartle 22.16,22.17) A four-dimensional spacetime has the line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right) dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{3}$$

(a) Find the Christoffel symbols for this metric by whatever method.

(b) Show that the Ricci curvature of this metric vanishes.