MATH4105 General Relativity Problem Sheet 7

All these problems from Carroll Chapter 5.

I. QUESTIONS

1/ Consider a particle (not necessarily on a geodesic) that has fallen inside the event horizon, $r < 2GM$. Use the ordinary Scwarzschild coordinates (t, r, θ, ϕ). Show that the radial coordinate must decrease at a minimum rate given by

$$\left|\frac{dr}{d\tau}\right| \geq \sqrt{\frac{2GM}{r} - 1} \tag{1}$$

Calculate the maximum lifetime for a particle along a trajectory from $r = 2GM$ to $r = 0$. Express this in seconds for a black hole with mass measured in solar masses. (You might need to look up an integral in a table or use Mathematica for it.)

2/ It can be shown that a general spherically symmetric metric has the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2}$$

The non-zero entries in the Ricci tensor turn out to be

$$\begin{aligned}
R_{tt} &= e^{2(\alpha-\beta)}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha\right] \\
R_{rr} &= -\partial_r^2\alpha - (\partial_r\alpha)^2 + \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\beta \\
R_{\theta\theta} &= e^{-2\beta}\left[r(\partial_r\beta - \partial_r\alpha) - 1\right] + 1 \\
R_{\phi\phi} &= \sin^2\theta R_{\theta\theta}
\end{aligned}$$

Consider Einstein's equations of general relativity with a cosmological constant

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0. \tag{3}$$

(where Λ is the cosmological constant.)

(a) Solve for the most general spherically symmetric metric that reduces to the Schwarzschild co-ordinates when $\Lambda = 0$. (Hint: Read Carroll section 5.1 before attempting this, you can follow each of the steps he uses to derive the Schwarzschild metric.)

(b) Write down the equation of motion for radial geodesics in terms of an effective potential as we did in lectures for the case when ($\Lambda = 0$). Sketch the effective potential for massive particles.

(c) By thinking about the orbit of Neptune for example, try to estimate an upper bound on the cosmological constant resulting from the fact that $\Lambda = 0$ gives an accurate description of planetary orbits.

3. Consider an observer sitting at constant spatial co-ordinates r_*, θ_*, ϕ_* around a Schwarzschild black hole of mass M. The observer drops a beacon into the black hole, (straight down, along a radial trajectory). The beacon emits radiation at a constant wavelength $\lambda_{\rm em}$ (in the beacon rest frame).

(a) Calculate the co-ordinate speed dr/dt of the beacon, as a function of r.

(b) Calculate the proper speed of the beacon. That is, the speed measured by an observer stationary at distance r from the black hole. What is this speed as $r \to 2GM$.

(c) Calculate the wavelength $\lambda_{\rm obs}$, measured by the observer at r_*, as a function of the radius $r_{\rm em}$ at which the radiation was emitted.

(d) Calcualte the time $t_{\rm obs}$ at which a beam emitted by the beacon at radius $r_{\rm em}$ will be observed at r_*.

(e) Show that at late times, the redshift grows exponentially: $\lambda_{\rm obs}/\lambda_{\rm em} \propto e^{t_{\rm obs}/T}$. (This requires that you make an approximation that is valid at very late co-ordinate times as beacon approaches the event horizon at $r = 2GM$). Give an expression for the time constant T in terms of the black hole mass M.