General Relativity 2008 Problem Sheet 7 Answers

1

1/ Consider a particle (not necessarily on a geodesic) that has fallen inside the event horizon, $r < 2GM$. Use the ordinary Scwarzschild coordinates (t, r, θ, ϕ). Show that the radial coordinate must decrease at a minimum rate given by

$$\left|\frac{dr}{d\tau}\right| \geq \sqrt{\frac{2GM}{r} - 1} \tag{1}$$

Calculate the maximum lifetime for a particle along a trajectory from $r = 2GM$ to $r = 0$. Express this in seconds for a black hole with mass measured in solar masses. (You might need to look up an integral in a table or use Mathematica for it.)

The particle has to be timelike so we have

$$-1 = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -\left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left[\left(\frac{dr}{d\tau}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right]. \tag{2}$$

Rearranging we find (recall that $r < 2GM$)

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{2GM}{r} - 1 + \left(1 - \frac{2GM}{r}\right)^2\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{2GM}{r} - 1\right)r^2\left[\left(\frac{dr}{d\tau}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right] \geq \frac{2GM}{r} - 1 \tag{3}$$

Where the inequality follows since all the terms left out are positive. Taking square roots gives the value

$$\left|\frac{dr}{d\tau}\right| \geq \sqrt{\frac{2GM}{r} - 1} \tag{4}$$

The maximum lifetime will occur when r changes as slowly as possible so it will saturate this inequality.

Looking up an integral

$$\int \frac{dx}{\sqrt{\frac{a}{x} - 1}} = \frac{\sqrt{x}(x - a) + a\sqrt{a - x}\tan^{-1}\left(\frac{\sqrt{x}}{\sqrt{a-x}}\right)}{\sqrt{\frac{a}{x} - 1}\sqrt{x}} \tag{5}$$

This gives

$$\begin{aligned}
\tau_{\max} &= \int_0^{2GM} \frac{dr}{\sqrt{\frac{2GM}{r} - 1}} = \left.\frac{\sqrt{r}(r - 2GM) + 2GM\sqrt{2GM - r}\tan^{-1}\left(\frac{\sqrt{r}}{\sqrt{2GM-r}}\right)}{\sqrt{\frac{2GM}{r} - 1}\sqrt{r}}\right|_0^{2GM} \\
&= \left.\frac{\sqrt{r}(r - 2GM) + 2GM\sqrt{2GM - r}\tan^{-1}\left(\frac{\sqrt{r}}{\sqrt{2GM-r}}\right)}{\sqrt{2GM - r}}\right|_0^{2GM} \\
&= \left.\sqrt{r}\sqrt{(r - 2GM} + 2GM\tan^{-1}\left(\frac{\sqrt{r}}{\sqrt{2GM - r}}\right)\right|_0^{2GM} = \pi GM
\end{aligned}$$

Clearly the function is zero at $r = 0$. At the upper limit we have to take a limit as $r \to 2GM$ (from below).

Recall that GM/c^2 is a length so putting the units back in we get (when M is the mass of the sun)

$$\tau_{\max} = \frac{\pi GM}{c^3} = \frac{\pi(6.673 \times 10^{-11})(1.989 \times 10^{30})}{(2.998 \times 10^8)^3} = 4.93\ \mu\text{s} \tag{6}$$

So for every solar mass of the black hole you have about five microseconds.

2

It can be shown that a general spherically symmetric metric has the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{7}$$

The non-zero entries in the Ricci tensor turn out to be

$$\begin{aligned}
R_{tt} &= e^{2(\alpha-\beta)}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha\right] \\
R_{rr} &= -\partial_r^2\alpha - (\partial_r\alpha)^2 + \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\beta \\
R_{\theta\theta} &= e^{-2\beta}\left[r(\partial_r\beta - \partial_r\alpha) - 1\right] + 1 \\
R_{\phi\phi} &= \sin^2\theta R_{\theta\theta}
\end{aligned}$$

Consider Einstein's equations of general relativity with a cosmological constant

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0. \tag{8}$$

(where Λ is the cosmological constant.)

(a) Solve for the most general spherically symmetric metric that reduces to the Schwarzschild co-ordinates when $\Lambda = 0$. (Hint: Read Carroll section 5.1 before attempting this, you can follow each of the steps he uses to derive the Schwarzschild metric.)

(b) Write down the equation of motion for radial geodesics in terms of an effective potential as we did in lectures for the case when ($\Lambda = 0$). Sketch the effective potential for massive particles.

(c) By thinking about the orbit of Neptune for example, try to estimate an upper bound on the cosmological constant resulting from the fact that $\Lambda = 0$ gives an accurate description of planetary orbits.

This one involved one or two tricks which were found in Carroll as I suggested, or else some pretty long calculation. The following is only one way of getting to the answer.

(a) We are given

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0 \tag{9}$$

and the values of $g_{\mu\nu}$ and $R_{\mu\nu}$.

The first trick is to get rid of $R = R^\mu_\mu$. We can use the inverse of the metric to raise one of the indices and obtain the equation

$$R^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu R + \Lambda\delta^\mu{}_\nu = 0 \tag{10}$$

Recall that $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$ and so in multiplying them we just get the identity. When we contract we find $\delta^\mu{}_\mu = 4$ and so we have

$$R - 2R + 4\Lambda = 0 \tag{11}$$

or $R = 4\Lambda$.

Substituting this into the field equations with one raised and one lowered index and rearranging we find that the field equations are

$$R^\mu{}_\nu = \Lambda\delta^\mu{}_\nu. \tag{12}$$

Now since the metric is just diagonal we have $g^{00} = -e^{-2\alpha}$ and $g^{11} = e^{-2\beta}$ and so we find

$$\begin{aligned}
R^0{}_0 &= g^{00}R_{00} = e^{-2\beta}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha\right] && (13) \\
R^1{}_1 &= g^{11}R_{11} = e^{-2\beta}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta - \frac{2}{r}\partial_r\beta\right] && (14)
\end{aligned}$$

Notice that these are almost identical so we have

$$R^0{}_0 - R^1{}_1 = e^{-2\beta}\left[\frac{2}{r}\partial_r\alpha + \frac{2}{r}\partial_r\beta\right] \tag{15}$$

One the other hand from the field equations we have

$$R^0{}_0 - R^1{}_1 = \Lambda(\delta^0{}_0 - \delta^1{}_1) = 0. \tag{16}$$

This gives us that

$$\partial_r\beta = -\partial_r\alpha \tag{17}$$

or

$$\beta = -\alpha + c \tag{18}$$

At this point we can actually rescale time to get rid of c but lets keep it around.

Now it's a big question what to do! The best thing appears to be to consider the 22-component of the Einstein equation. Going back to lowered indices this is

$$R_{22} = \Lambda g_{22} = r^2\Lambda \tag{19}$$

On the other hand we are given

$$R_{22} = e^{-2\beta}\left[r(\partial_r\beta - \partial_r\alpha) - 1\right] + 1 \tag{20}$$

By using our identity for $\partial_r\alpha$ we find

$$e^{-2\beta}\left[1 - 2r\partial_r\beta\right] = 1 - r^2\Lambda. \tag{21}$$

This differential equation for β is actually pretty easy to solve if you notice that

$$e^{-2\beta}\left[1 - 2r\partial_r\beta\right] = \partial_r\left[re^{-2\beta}\right]. \tag{22}$$

As a result we can directly integrate the left and right hand sides to find

$$re^{-2\beta} = r - r^3\Lambda/3 + d \tag{23}$$

where d is another integration constant. This gives us

$$g_{11} = e^{2\beta} = \left(1 + \frac{d}{r} - \Lambda r^2/3\right)^{-1} \tag{24}$$

Agreement with the Schwarzschild metric when $\Lambda = 0$ gives $d = -2GM$. We also have

$$g_{11} = e^{2\alpha} = e^{-2c}e^{-2\beta} = e^{-2c}\left(1 - \frac{2GM}{r} - \Lambda r^2/3\right). \tag{25}$$

Once again agreement with Schwarzschild gives $c = 0$ and we have determined both the unknown functions in the metric.

b) Here we just step through the procedure in the lectures to find the orbits of particles in the new metric. Once again the metric is independent of ϕ and so we find that the dynamics preserves an angular momentum per unit mass

$$l = r^2\frac{d\phi}{d\tau}. \tag{26}$$

Likewise the metric is independent of time so it has an energy per unit mass

$$e = \left(1 - \frac{2GM}{r} - \Lambda r^2/3\right)\frac{dt}{d\tau}. \tag{27}$$

We see that a particle moving in the x-z plane having $\phi = 0$ and $d\phi/d\tau(0) = 0$ will stay in that plane so once again the orbits are confined to planes. Choosing that plane now to be the xy-plane and so $\sin\theta = 0$ and $d\theta/d\tau = 0$ we may substitute into the usual formula for the normalisation of the four-velocity for a timelike particle to find

$$-1 = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = \left(1 - \frac{2GM}{r} - \Lambda r^2/3\right)^{-1}\left[-e^2 + \left(\frac{dr}{d\tau}\right)^2\right] + \frac{l^2}{r^2} \tag{28}$$

Rearranging the equations as before gives the orbit equation

$$\frac{1}{2}\left(e^2 - 1 + \Lambda l^2/3\right) = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}}(r) \tag{29}$$

where we have collected all of the r dependent terms in the effective potential

$$V_{\text{eff}}(r) = -\frac{GM}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{r^3} - \Lambda r^2/6. \tag{30}$$

We see that for large radii the potential is affected by the Λ term. Depending on the sign objects are either always bound or they eventually experience repulsive forces leading away from the centre.

Notice that the term proportional to Λ is leads to a central force proportional to the distance. Newton proved that this and the $1/r^2$ law were the only two where a spherical mass distribution leads to a force that is the same as if the mass were collected at a single point.

Since the orbit of Neptune say is pretty well described by $\Lambda = 0$ we conclude that the term proportional to Λ in the metric is very small so that

$$\Lambda r^2/3 \ll 2GM/r \tag{31}$$

which implies that

$$\Lambda \ll 6GM/r^3 \tag{32}$$

where r^3 could be the radius of the orbit of Neptune and M the mass of the sun. It turns out that this places a very weak bound on the cosmological constant, one that gives an upper bound many orders of magnitude smaller than the apparent value. (Notice that if we convert Λ into a mass/energy density $\Lambda = -8\pi G\rho_\Lambda$ then this bound is just that the energy density due to the cosmological constant is much less than the energy density due to averaging the mass M over the volume of the sphere of radius r: $4\pi r^3/3$)

$$\rho_\Lambda \ll 3M/4\pi r^3 \simeq 5\times 10^{-9}\ \text{kg/m}^3 \tag{33}$$

The the dark energy density is thought to be very roughly 10^{-27} kg/m^3 which gives an idea of how hard it would be to measure directly, without looking at effects on cosmological scales.

3

Consider an observer sitting at constant spatial co-ordinates r_*, θ_*, ϕ_* around a Schwarzschild black hole of mass M. The observer drops a beacon into the black hole, (straight down, along a radial trajectory). The beacon emits radiation at a constant wavelength λ_{em} (in the beacon rest frame).

(a) Calculate the co-ordinate speed dr/dt of the beacon, as a function of r.

(b) Calculate the proper speed of the beacon. That is, the speed measured by an observer stationary at distance r from the black hole. What is this speed as $r \to 2GM$.

(c) Calculate the wavelength λ_{obs}, measured by the observer at r_*, as a function of the radius r_{em} at which the radiation was emitted.

(d) Calcualte the time t_{obs} at which a beam emitted by the beacon at radius r_{em} will be observed at r_*.

(e) Show that at late times, the redshift grows exponentially: $\lambda_{\text{obs}}/\lambda_{\text{em}} \propto e^{t_{\text{obs}}/T}$. (This requires that you make an approximation that is valid at very late co-ordinate times as beacon approaches the event horizon at $r = 2GM$). Give an expression for the time constant T in terms of the black hole mass M.

a) The beacon is on a radial geodesic and we can apply the results of the orbit theory discussed in class.

We take the initial instant to have both co-ordinate time and proper time equal to zero. We choose the particle to be in the xy-plane as in lectures. The particle is initially at rest so that all spatial proper time derivatives are zero. The initial conditions are

$$x^\mu = (0, r^*, \pi/2, 0) \tag{34}$$

and

$$\left.\frac{dx^\mu}{d\tau}\right|_{\tau=0} = \left(\left.\frac{dt}{d\tau}\right|_{\tau=0}, 0, 0, 0\right). \tag{35}$$

Normalisation of u^μ requires that

$$\left.\frac{dx^\mu}{d\tau}\right|_{\tau=0} = \left(\frac{1}{\sqrt{1-\frac{2GM}{r^*}}}, 0, 0, 0\right). \tag{36}$$

This gives that value of the energy and angular momentum valid for the whole geodesic

$$e_b = \left(1 - \frac{2GM}{r^*}\right)\left.\frac{dt}{d\tau}\right|_{\tau=0} = \sqrt{1-\frac{2GM}{r^*}} \tag{37}$$

$$l_b = r^{*2}\left.\frac{d\phi}{d\tau}\right|_{\tau=0} = 0 \tag{38}$$

We can find the $dr/d\tau$ as a function of r by substituting into the normalisation condition

$$-1 = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -\left(1-\frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 \tag{39}$$

We find

$$\left(\frac{dr}{d\tau}\right)^2 = e_b^2 - \left(1-\frac{2GM}{r}\right) = \frac{2GM}{r} - \frac{2GM}{r^*} \tag{40}$$

Notice that this velocity starts from zero and grows without bound as the singularity approaches. Notice however that it is finite and non-zero at the horizon. The beacon is moving very fast here though, if it started a long way from the horizon.

And finally

$$\left(\frac{dr}{dt}\right)^2 = \left(\frac{dr}{d\tau}\right)^2\left(\frac{d\tau}{dt}\right)^2 = \left(\frac{2GM}{r} - \frac{2GM}{r^*}\right)\frac{\left(1-\frac{2GM}{r}\right)^2}{e_b^2} = \left(\frac{2GM}{r} - \frac{2GM}{r^*}\right)\frac{r^*\left(r-2GM\right)^2}{r^2\left(r^*-2GM\right)} \tag{41}$$

This velocity starts from zero at r^* as before but approaches zero as the particle approaches the horizon. This is because $dt/d\tau$ is approaching infinity here. This is telling us that in order to track the motion of the particle we really should change to a set of co-ordinates that is not singular here.

To find the velocities we take the square roots and since the beacon is ingoing we should have the negative square root.

b) Recall that to find the measured energy for an observer with four velocity $u^\mu_{\rm obs}$ we evaluate $-g_{\mu\nu}u^\mu_{\rm obs}p^\nu$ which is a perfectly fine scalar. The four velocity functions as a unit vector in the time direction and thus allows us to write down the measured value of the time component of four-vectors.

To work out the velocity by a stationary observer at r we need a similar formula for the velocity. The basic idea is that the time component of the four velocity of the particle. In special relativity we have $dt/d\tau = \gamma = 1/\sqrt{1-|\vec{v}|^2}$ where $\vec{v}$ is the measured velocity for an observer at rest. The four velocity for that observer is of course $\mathbf{u}_{\rm obs} = (1, 0, 0, 0)$ so we can write

$$\gamma = -\mathbf{u}_{\rm obs}\cdot\mathbf{u}. \tag{42}$$

So we find, see the study notes for more,

$$|\vec{v}| = \sqrt{1-(-u_{{\rm obs}\mu}u^\mu)^{-2}}. \tag{43}$$

where $u_{\text{obs}\mu}$ is the four-vector velocity of our observer and u^μ is the four-vector velocity of the beacon.

From the previous question the four velocity of the beacon at r is

$$u^\mu = \left(\frac{\sqrt{1-\frac{2GM}{r^*}}}{1-\frac{2GM}{r}}, -\sqrt{\frac{2GM}{r} - \frac{2GM}{r^*}}, 0, 0 \right). \tag{44}$$

While the four velocity of the observer who is at rest at r is

$$u^\mu_{\text{obs}} = \left(\frac{1}{\sqrt{1-\frac{2GM}{r}}}, 0, 0, 0 \right). \tag{45}$$

We find

$$|v| = \sqrt{1 - \frac{1-\frac{2GM}{r}}{1-\frac{2GM}{r^*}}} \tag{46}$$

Notice that the velocity approaches the speed of light as the beacon approaches the horizon. Despite the fact that the beacon is moving slowly in our co-ordinate system, physically the relative velocity of the beacon and an observer at rest approaches the speed of light.

c) This one we can calculate following exactly the same steps as in lectures for two observers at rest in the co-ordinate system, using the fact that the light ray has a conserved energy. It's slightly complicated here by the fact the beacon is receding from the stationary observer at a speed that can be a large fraction of the speed of light, so the redshift will be larger.

From the previous questions the four velocity of the beacon at r_{em} is

$$u^\mu = \left(\frac{\sqrt{1-\frac{2GM}{r^*}}}{1-\frac{2GM}{r_{\text{em}}}}, -\sqrt{\frac{2GM}{r_{\text{em}}} - \frac{2GM}{r^*}}, 0, 0 \right) \tag{47}$$

The light ray has will be a radial light ray having

$$\frac{dx^\mu}{d\tau} = \left(\frac{dt}{d\lambda}, \frac{dr}{d\lambda}, 0, 0 \right). \tag{48}$$

Now recall that the light is a null particle so that

$$g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0. \tag{49}$$

If the particle was massive it would have $p^\mu = m_0 dx^\mu/d\tau$ as its four-vector momentum but light does not have a rest mass. For a suitable choice of the parameter λ the four-vector momentum is just equal to the four-vector velocity $p^\mu = dx^\mu/d\tau$, see my notes on special rel. The energy of a photon observed at r_{em} is $-u_{\text{em}\mu} p^\mu(r_{\text{em}})$.

For the observer at r^* the four velocity is the same as the initial four-velocity of the beacon discussed in a).

$$u^\mu_{\text{obs}} = \left(\frac{1}{\sqrt{1-\frac{2GM}{r^*}}}, 0, 0, 0 \right) \tag{50}$$

For the light ray just as for the beacon there is a conserved energy as a result of its motion along a geodesic which we will label e_l. We have

$$e_l = \left(1 - \frac{2GM}{r}\right) \frac{dt}{d\lambda}. \tag{51}$$

We also have the null condition on the four-velocity for a radial light ray which is

$$0 = g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = \left(1 - \frac{2GM}{r}\right) \left(\frac{dt}{d\lambda}\right)^2 - \left(1 - \frac{2GM}{r}\right)^{-1} \left(\frac{dr}{d\lambda}\right)^2. \tag{52}$$

This implies on rearranging that

$$\frac{dr}{d\lambda} = \pm e_l \tag{53}$$

with the positive square root being appropriate since the light beam is being sent back from the beacon to the original observer. We've found a much more convenient representation for the four-velocity of the photon as a function of r

$$p^\mu = \frac{dx^\mu}{d\lambda} = \left(\frac{e_l}{1 - \frac{2GM}{r}}, e_l, 0, 0 \right). \tag{54}$$

since e_l is a constant independent of r.

Now we just need to find the photon energies measured by the two observers. We can start with the fixed observer at r^*, we find

$$E_{\text{obs}} = -u_{\text{obs}\mu} p^\mu(r^*) = \left(1 - \frac{2GM}{r^*}\right) \frac{1}{\sqrt{1 - \frac{2GM}{r^*}}} \frac{e_l}{1 - \frac{2GM}{r^*}} = \frac{e_l}{\left(1 - \frac{2GM}{r^*}\right)^{1/2}} \tag{55}$$

Notice that this is perfectly in agreement with our calculation for stationary observers in class.

Performing the same calculation for the beacon gives

$$E_{\text{em}} = -u_{\text{em}\mu} p^\mu(r_{\text{em}}) = \left(1 - \frac{2GM}{r_{\text{em}}}\right) \frac{\sqrt{1 - \frac{2GM}{r^*}}}{1 - \frac{2GM}{r_{\text{em}}}} \frac{e_l}{1 - \frac{2GM}{r_{\text{em}}}} - \left(1 - \frac{2GM}{r_{\text{em}}}\right)^{-1} \left(-\sqrt{\frac{2GM}{r_{\text{em}}} - \frac{2GM}{r^*}} \right) e_l \tag{56}$$

Now the redshift is

$$\frac{\lambda_{\text{obs}}}{\lambda_{\text{em}}} = \frac{E_{\text{em}}}{E_{\text{obs}}} = \frac{\sqrt{1 - \frac{2GM}{r^*}}}{1 - \frac{2GM}{r_{\text{em}}}} \left(\sqrt{1 - \frac{2GM}{r^*}} + \sqrt{\frac{2GM}{r_{\text{em}}} - \frac{2GM}{r^*}} \right)$$

This expression is quite different from that of a stationary beacon because of the relative velocity of the emitter and the sender which is becoming very large near the horizon.

d) Consider again a radial light path. This has

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 = 0 \tag{57}$$

This gives

$$\frac{dr}{d\lambda}\frac{d\lambda}{dt} = \frac{dr}{dt} = 1 - \frac{2GM}{r} \tag{58}$$

We find that the proper time for this path is given by solving the integral

$$\int dt = t + c = \int \frac{dr}{1 - \frac{2GM}{r}} = r + 2GM \log(r - 2GM) \tag{59}$$

Notice that this integral is easier than it looks since

$$\frac{1}{1 - \frac{2GM}{r}} = 1 + \frac{2GM}{r - 2GM} \tag{60}$$

Suppose that at $t = t_{\text{em}}$ we have $r = r_{\text{em}}$ then we get

$$c = r_{\text{em}} - t_{\text{em}} + 2GM \log(r_{\text{em}} - 2GM) \tag{61}$$

and the arrival time at $r = r^*$ is

$$t_{\text{obs}} - t_{\text{em}} = (r^* - r_{\text{em}}) + 2GM \log \frac{r^* - 2GM}{r_{\text{em}} - 2GM} \tag{62}$$

This propagation "time" gets to be very long as the beacon gets close to the event horizon. Notice that the first term is just the propagation time expected in a flat space time and the second is the "delay" that light recieves in moving close to a massive object. This is the term that led to the "Shaprio delay" for light signals sent past the sun, which we discussed in class.

e) We want to find the time dependence of the redshift observed at r^*. At the moment we only have the redshift as a function of $r_{\rm em}$. We would like to find it in terms of the observation time $t_{\rm obs}$. To do this we need to know how the beacon's location changes with time in order to find $t_{\rm em}$ and then use the results of d) to find $t_{\rm obs}$. (We would still get an exponential answer if we wrote it in terms of $t_{\rm em}$ the co-ordinate time at emission. But we would have a different co-efficient on the exponential.)

We are allowed to work in the limit in which $r_{\rm em} \simeq 2GM$ so we will set $r_{\rm em}(t) \simeq 2GM + \delta r$ and we will rescale time so that $t = 0$ occurs for some value of δr such that $\delta r/2GM \ll 1$ and only keep terms up to lowest order in this small parameter. We will also assume that the observer at r^* is very far from the horizon at $2GM$ so that $r^* - 2GM \simeq r^*$ and $2GM/r^* \ll 1$.

Firstly lets consider the expression for dr/dt from a)

$$\frac{d\delta r}{dt} = -\sqrt{\left(\frac{2GM}{2GM+\delta r} - \frac{2GM}{r^*}\right)\frac{r^*(\delta r)^2}{(2GM+\delta r)^2\,(r^*-2GM)}} \simeq -\frac{\delta r}{2GM} \tag{63}$$

Notice that this leading order time derivative is indeed small given our assumptions. Our assumptions would not be consistent if this was not the case.

This equation is easily integrated to give

$$\delta r \simeq \delta r(0) e^{-t/2GM}. \tag{64}$$

We want to perform the same expansion in the formula for the redshift. We can use

$$\frac{2GM}{r_{\rm em}} - \frac{2GM}{r^*} \simeq 1 - \frac{2GM}{r^*} \tag{65}$$

and

$$\frac{1}{1-\frac{2GM}{r_{\rm em}}} = 1 + \frac{2GM}{r_{\rm em}-2GM} = 1 + \frac{2GM}{\delta r} \tag{66}$$

to get the approximation to our answer in c)

$$\frac{\lambda_{\rm obs}}{\lambda_{\rm em}} \simeq \left(1-\frac{2GM}{r^*}\right)\frac{4GM}{\delta r} \tag{67}$$

Finally consider the observation time in terms of δr from our answer to part d)

$$t_{\rm obs} = (r^* - 2GM - \delta r) + t_{\rm em} - 2GM\log\frac{\delta r}{r^*\left(1-\frac{2GM}{r^*}\right)} \tag{68}$$

This is the exact expression.

We can substitute for our approximate expression for $t_{\rm em}$ in terms of δr

$$t_{\rm em} \simeq -2GM\log[\delta r/\delta r(0)] \tag{69}$$

to find

$$t_{\rm obs} = (r^* - 2GM - \delta r) - 2GM\log\frac{\delta r}{\delta r(0)} - 2GM\log\frac{\delta r}{r^*\left(1-\frac{2GM}{r^*}\right)} \tag{70}$$

$$= (r^* - 2GM - \delta r) - 4GM\log\frac{\delta r}{\delta r(0)} - 2GM\log\frac{\delta r(0)}{r^*\left(1-\frac{2GM}{r^*}\right)}. \tag{71}$$

Notice how we have used the usual rules for logarithms to get from the first line to the second.

This expression give the observation time $t_{\rm obs}$ in terms of δr and various terms that are constant. For sufficiently small $\delta r/\delta r(0)$ it is the middle term that dominates and we can ignore the term proportional to δr and the constant terms. This will hold after long enough times and we find in this limit

$$\delta r \simeq \delta r(0) e^{-t_{\rm obs}/4GM} \tag{72}$$

for sufficiently large t that the exponential is very small. We can use this expression to replace δr in the approximate expression for the redshift to obtain

$$\frac{\lambda_{\rm obs}}{\lambda_{\rm em}} \simeq \left(1 - \frac{2GM}{r^*}\right) \frac{4GM}{\delta r(0)} e^{t_{\rm obs}/4GM} \tag{73}$$

which is the desired exponential time dependence. We have

$$4GM/c^3 \simeq 2.0 \times 10^{-5} \left(\frac{M}{M_\odot}\right) s \tag{74}$$

so that this change in the frequency of the emitted light is incredibly rapid even for a modest size black hole.

(In his Ex12.2 Hartle manages to get the time constant $4GM$ by another method using Eddington-Finkelstein co-ordinates.)