# MATH4105 General Relativity Problem Sheet 8 Background

We're going to try and understand some features of the current cosmological model.

The currently accepted values of the model are given in a review article for the Review of Particle Physics [1], although the WMAP collaboration has since published new data. It is conventional to write the Hubble constant as $H_0 = h \times 100$ km/s/Mpc. Also mass densities are often quoted in terms of the critical energy density which would give a flat universe

$$\rho_{\text{crit}} = \frac{3H_0^2}{8\pi G}$$

So that if $\rho_b$ is the energy density of baryons (think protons and neutrons, the mass of electrons can be ignored, there are as many electrons as there are protons though) then

$$\Omega_b = \frac{\rho_b}{\rho_{\text{crit}}}.$$

All the energy densities are assumed to be constant on the cosmological scales we are interested in.

With this in mind the preferred values for the current cosmological parameters are

| | | | |
|---|---|---|---|
| Hubble parameter | $h$ | $0.73 \pm 0.03$ | (1) |
| Total matter density | $\Omega_m$ | $\Omega_m h^2 = 0.128 \pm 0.008$ | (2) |
| Baryon density | $\Omega_b$ | $\Omega_b h^2 = 0.0223 \pm 0.0007$ | (3) |
| Radiation density | $\Omega_r$ | $\Omega_r h^2 = 2.47 \times 10^{-5}$ | (4) |

The radiation density is simply the photons in the cosmic microwave background.

The universe also contains neutrinos but their mass is very uncertain and it is essentially assumed that their energy density is always negligible.

The matter density is largely made up of unknown particles known as dark matter $\Omega_m = \Omega_b + \Omega_{dm}$. Dark matter is likely to be made up of particles that do not interact very strongly (like neutrinos with more mass) and therefore fell out of equilibrium with the rest of the universe very early in its evolution. The dark matter is assumed to be made up of particles heavy enough that they are moving at non-relativistic speeds even in the very early universe and therefore have a negligible pressure. Hence it is often known as cold dark matter. At the present time the baryons can also be assumed to have zero pressure (zero temperature ideal gas effectively.)

We will assume that the universe is flat and that the missing energy comes from an energy density $\rho_\Lambda$ with $p_\Lambda = -\rho_\Lambda$ called variously dark energy or the cosmological constant. So we have

$$\Omega_r + \Omega_m + \Omega_\Lambda = 1 \tag{5}$$

This means that the metric for the universe is given by

$$ds^2 = dt^2 - a(t)^2[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{6}$$

with $a(t)$ given by the Friedman equations.

All of the components of the universe are now out of equilibrium so the energy conservation equation

$$\frac{d}{dt}(\rho a^3) = -3a^2 p \tag{7}$$

applies to each source of energy density separately. At early times the baryons and photons were in thermal equilibrium and so we should consider energy conservation for the two gases together.

The baryons in the universe are currently made up of hydrogen (75%) and helium (25%) nuclei and are sufficently dilute that light propagates without scattering. However in the early universe the nuclei, electrons and photons formed a high temperature plasma that was opaque to light. (The plasma was opaque since electrons scatter light very strongly in a process known as Thompson scattering, this is the low energy limit of Compton scattering. Atoms on the other hand do not scatter light nearly as strongly.) As the universe expanded the plasma became more dilute and light began to propagate as its mean free path increased (*decoupling*) and as the temperature dropped atoms formed out of the ionised nuclei and electrons (*recombination*). Both effects led to a sudden drop in the scattering of light by electrons. The light was then free to propagate to us, and the matter and radiation fell out of equilibrium.

(At some later stage star light ionised most of the atoms again but this need not concern us since the energy densities were then too low for for the light scattering that resulted to be significant.)

It's possible to work out roughly the temperature of the universe at recombination. We will do this in the problem. It is a simple application of chemical kinetics. Think of the ionisation of hydrogen as a chemical reaction

$$p + e \rightarrow H. \tag{8}$$

The ionisation of hydrogen is described by the Saha equation (this is the chemical kinetics equation for this special case)

$$\frac{n_e}{n_p} n_H = \left( \frac{m_e k T}{2\pi \hbar^2} \right)^{3/2} e^{-E_I/kT} \tag{9}$$

where $E_I$ is the ionisation energy of hydrogen (13.6 eV) and $n_e$ is the number density of electrons. (This equation ignores the presence of a large amount of light which modifies the ionisation temperature slightly).

As a result of all this the radiation in the universe has a black body distribution at temperature $T = 2.725$. It might help to recall that the energy density of a black body at temperature $T$ is

$$\rho = \sigma T^4 \tag{10}$$

where $\sigma = 8\pi^5 k_B^4 / 15 h^3 c^3$. (Sorry $h$ in this formula is Planck's constant, I won't use it anywhere else.) Also the number density of photons is

$$n = 3.7 \frac{\sigma T^3}{k_B}. \tag{11}$$

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[1] O. Lahav and A. R. Riddle (2008), http://pdg.lbl.gov/2008/reviews/hubblerpp.pdf.