General Relativity Study Notes

These notes attempt to summarize the overall formalism, assuming that you have already seen things like the Einstein summation convention and the occurence of raised and lowered indices.

The basic assumption of general relativity is that times and distances are described by a pseudo-Riemannian metric $g_{\mu\nu}$. And that, in the abscence of forces other than gravity, particles follow geodesics of the geometry. The metric itself is determined by the distribution of energy (and pressure) through Einstein's equation.

The fact that space-time is curved means that we have to be very careful that the quantities we write down are independent of the co-ordinate system that we are working in.

I. PARTICLE TRAJECTORIES, LENGTHS AND TIMES

Here are some facts that work in all geometries. We will consider a particle path in space-time described by the paramter λ $x^\mu(\lambda)$ where μ runs over the four space-time indices. Notice that we will use a variety of co-ordinates x^μ in this course t, r, θ, ϕ for example in spherical polar co-ordinates.

We may define the four-vector $dx^\mu/d\lambda$. If

$$g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} < 0 \tag{1}$$

then the particle trajectory is *timelike.* All actual physical trajectories of massive particles are timelike.

For timelike trajectories it makes sense to ask how much time a clock moving along with the particle would advance during the motion. This quantity is usually known as the "proper time of the particle" and the assumption is that this is given us by the metric

$$\tau = \int_{\lambda_0}^{\lambda_1} d\lambda \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}. \tag{2}$$

On the other hand if

$$g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} > 0 \tag{3}$$

then the particle trajectory is *spacelike.* In this case the particle trajectory can be assigned a physical length. This length is likewise given by the metric

$$s = \int_{\lambda_0}^{\lambda_1} d\lambda \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}. \tag{4}$$

Finally if

$$g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0 \tag{5}$$

the particle trajectory is null. Light and other massless particles travel null trajectories.

II. FOUR-VECTOR VELOCITIES AND MOMENTA

We have not yet specified properties of the parameter λ. The timelike geodesics of the metric are most easily specified when the particle trajectory is parameterized by the proper time τ. For $\lambda_0 = 0$ if we can solve the integral Eq. (2) for the whole range of λ_1 we find $\tau(\lambda_1)$ and can in principle invert this function to write the position as a function of τ, $x^\mu(\tau)$. The resulting velocity four-vector is so interesting that we give it special letter and refer to it as *the* four-vector velocity

$$u^\mu = \frac{dx^\mu}{d\tau}. \tag{6}$$

The most important property of the four-velocity is that it is normalised. Consider the equation (for a trajectory starting at $\tau = 0$ and ending at $\tau = \tau'$)

$$\tau' = \int_0^{\tau'} d\tau \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}. \tag{7}$$

We may differentiate this with respect to the final proper time τ' to obtain (by using the fundamental theorem of calculus)

$$1 = \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\tau}(\tau') \frac{dx^\nu}{d\tau}(\tau')} = -g_{\mu\nu} \frac{dx^\mu}{d\tau}(\tau') \frac{dx^\nu}{d\tau}(\tau'). \tag{8}$$

Where the final equality follows because the value on the left hand side is one. Using the convention that the metric "lowers" an index we may write this equation in the shorthand

$$u_\mu u^\mu = -1. \tag{9}$$

Now our particle will have mass. This mass can be measured by testing the response of a particle at rest to a force, we will call it m_0. It is then common do define a four-momentum in analogy with classical mechanics

$$p^\mu = m_0 u^\mu. \tag{10}$$

For null trajectories we cannot improve on $dx^\mu/d\lambda$ since there is no proper time for such a trajectory. We will defer a discussion of the appropriate momentum four-vector until we get to the special relativity section.

III. SCALAR PRODUCTS

In physics we are interested in quantities that are independent of the co-ordinate system we have used to describe our physical system. In general relativity this invariance is sometimes known as general covariance. It generalises the invariance under rotations and translations familiar from Newtonian mechanics, and under Lorentz transformations as in special relativity.

So for example we should bear in mind that the 4-vectors and tensors that appear in the theory have a co-ordinate independent existence.

Many of the most important quantities we are interested in can be written as scalar products of some kind. For two 4-vectors x^μ and y^μ the scalar product is

$$x \cdot y = g_{\mu\nu} x^\mu y^\nu = x_\mu y^\mu. \tag{11}$$

Scalar products of vectors do not depend on the co-ordinates we have chosen.

IV. CHANGE OF CO-ORDINATES

Speaking of changes of co-ordinates. If we change to a new set of co-ordinates $x'^\mu(x^\nu)$ then we must use expressions for the old co-ordinates in terms of the new and the chain rule in all our expressions to get rid of the old co-ordinates. In particular the metric changes according to

$$g_{\mu\nu}(x') = g_{\kappa\lambda}(x(x')) \frac{\partial x^\kappa}{\partial x'^\mu} \frac{\partial x^\lambda}{\partial x'^\nu}. \tag{12}$$

Check that this formula guarantees that the scalar product of two four-velocities is invariant under co-ordinate transformations

V. GEODESICS

In the absence of forces other than gravity our particle will follow a geodesic. The geodesics of a timelike path extremize the proper time. In special relativity and for most curved spacetimes, the proper time is in fact a local maximum. The calculus of variations allows us to follow these extremizing paths given initial values of x^μ and u^μ.

First lets introduce the notation

$$w\left(x^\mu, \frac{dx^\mu}{d\tau}\right) = g_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}. \tag{13}$$

Notice that we are interpreting the co-ordinates x^μ and velocities u^μ as *independent variables* for the purposes of determining the geodesic equations. The geodesics satisfy the Euler-Lagrange equations and these may be written in the form

$$\frac{d}{d\tau}\frac{\partial w}{\partial(dx^\mu/d\tau)} = \frac{\partial w}{\partial x^\mu}. \tag{14}$$

Notice that if the right hand side is zero then it is possible to find a quantity that is constant along geodesics by looking at the argument of the derivative on the left-hand-side.

It is also sometimes useful to have the formula

$$\frac{d^2x^\mu}{d\tau^2} = -\Gamma^\mu_{\kappa\lambda}\frac{dx^\kappa}{d\tau}\frac{dx^\lambda}{d\tau} \tag{15}$$

where the Christoffel symbols are given by

$$\Gamma_{\mu\nu\kappa} = \frac{1}{2}\left(\partial_\mu g_{\kappa\nu} + \partial_\nu g_{\kappa\mu} - \partial_\kappa g_{\mu\nu}\right) \tag{16}$$

and

$$\Gamma^\kappa_{\mu\nu} = g^{\kappa\lambda}\Gamma_{\mu\nu\lambda}. \tag{17}$$

Recall that the metric with raised indices is just the inverse of the lower index version

$$g^{\mu\kappa}g_{\kappa\nu} = \delta^\mu_{\ \nu} \tag{18}$$

VI. SPECIAL RELATIVITY

The simplest example is when

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{19}$$

The Christoffel symbols are all zero and we get that $u^\mu(\tau)$ is constant.

Now we may identify the components of u^μ with velocities and make contact with the traditional quantities of special relativity. Lets start with the velocities

$$v^i = \frac{dx^i}{dt} = \frac{dx^i}{d\tau}\frac{d\tau}{dt} \tag{20}$$

which we may write

$$\vec{v} = \frac{d\tau}{dt}\frac{d\vec{x}}{d\tau}. \tag{21}$$

Recall that Roman indices run over the spatial co-ordinates $i = x, y, z$ and $\vec{x}$ is the spatial three-vector obtained by lopping off the time-cordinate x^0. This is just *the velocity of the particle as measured by an observer at rest in our current co-ordinates* t, x, y, z. Normalisation gives

$$1 = u_\mu u^\mu = \left(\frac{dt}{d\tau}\right)^2 (1 - |v|^2) \tag{22}$$

or

$$\left(\frac{dt}{d\tau}\right) = \frac{1}{\sqrt{1-|v|^2}} = \gamma. \tag{23}$$

Where we have identified the usual gamma factor of special relativity. The velocity four-vector is then

$$u = \begin{pmatrix} \gamma \\ \gamma \vec{v} \end{pmatrix} \tag{24}$$

Notice that if we know the velocity four-vector in these co-ordinates we may find the magnitude of the velocity measured in this frame from the formula

$$|v| = \sqrt{1 - (u^0)^{-2}} \tag{25}$$

Now this is not a co-ordinate independent statement since a transformation on x^μ will change the four-vector velocity. Surely we can express this velocity in a co-ordinate independent fashion? We just need to recognise that velocity four-vector for our observer (at rest in these co-ordinates) is

$$u_{\text{obs}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{26}$$

Thus we can write the measured velocity as

$$|v| = \sqrt{1 - (u_{\text{obs}\mu} u^\mu)^{-2}}. \tag{27}$$

This formula involves only constants and a scalar product and as a result is independent of co-ordinates.

This formula generalizes to arbitrary curved space-times. In fact the basic assumption in general relativity, that everything looks locally like special relativity, requires that we generalise the equations in this way.

The momentum four-vector is then

$$p^\mu = \begin{pmatrix} \gamma m_0 \\ \gamma m_0 \vec{v} \end{pmatrix} \equiv \begin{pmatrix} E \\ \vec{p} \end{pmatrix}. \tag{28}$$

Where we have defined the energy E and momentum $\vec{p}$ in the way that is usual in special relativity. We have the useful identity

$$E^2 - |\vec{p}|^2 = p_\mu p^\mu = m_0^2 u_\mu u^\mu = m_0^2. \tag{29}$$

The four-momentum is the quantity that is conserved in collisions of particles in special relativity, unifying the separate Newtonian conservation of energy and momentum.

For small enough velocities, such that $\gamma \simeq 1 + |v|^2/2$, these expressions give

$$E \simeq m_0 + \frac{1}{2} m_0 |v|^2 \tag{30}$$

$$\vec{p} \simeq m_0 \vec{v} \tag{31}$$

which agree with the Newtonian expressions when the rest energy of a mass $E = mc^2$ is included.

Finally $E = p^0$ is the energy measured by an observer at rest in our co-ordinates with $u^\mu_{\text{obs}} = (1, 0, 0, 0)$. We can write this energy in co-ordinate independent form as

$$E = u_{\text{obs}\mu} p^\mu \tag{32}$$

Once again it is this formula that generalizes to curved spacetimes.

A. Light

A plane wave of light has electric field proportional to

$$e^{-i(\omega t - \vec{k} \cdot \vec{x} + \delta)} = e^{-i(k_\mu x^\mu + \delta)} \tag{33}$$

where we have defined the wave four vector

$$k = \begin{pmatrix} \omega \\ \vec{k} \end{pmatrix} \tag{34}$$

Since we can write the phase of the wave as a scalar product it is Lorentz invariant. Recall that $\omega = c|\vec{k}|$ so we have (in units where $c = 1$)

$$k_\mu k^\mu = \omega^2 - |k|^2 = \omega^2 - \omega^2 = 0. \tag{35}$$

So the wave four vector is a null vector.

The wavefronts move at the speed of light in the direction given by $\vec{k}$ and are described by the condition

$$k_\mu x^\mu + \delta = 2\pi n \tag{36}$$

Lets choose a point $x_s^\mu = (t_s, x_s, y_s, z_s)$ on the wavefront with $n = 1$ (that is $k_\mu x_s^\mu + \delta = 2\pi$). Consider the trajectory

$$x^\mu(\lambda) = \begin{pmatrix} \omega\lambda + t_s \\ k_1\lambda + x_s \\ k_2\lambda + y_s \\ k_3\lambda + z_s \end{pmatrix}. \tag{37}$$

Notice firstly that this point travels along with the wavefront since

$$k_\mu x^\mu(\lambda) + \delta = \lambda k_\mu k^\mu + k_\mu x_s^\mu + \delta = 2\pi. \tag{38}$$

Notice secondly that this point on the wave front moves at the speed of light in the direction of $\vec{k}$ since

$$v^i = \frac{dx^i}{d\lambda}\frac{d\lambda}{dt} = k^i/\omega \tag{39}$$

which is a vector $\vec{v} = \vec{k}/\omega$ having length one and pointing in the correct direction.

Notice finally that

$$\frac{dx^\mu}{d\lambda} = k^\mu \tag{40}$$

so the four-velocity for a plane wave of light, with this parameterization λ is just the wave four vector k^μ.

Now we can determine the momentum four vector for light most easily by recalling the quantum mechanical expressions for the energy and momentum of a photon

$$E = \hbar\omega \tag{41}$$

$$\vec{p} = \hbar\vec{k} \tag{42}$$

or in four vector notation

$$p^\mu = \hbar k^\mu. \tag{43}$$

Notice that we may parameterize our wavefront trajectory in terms of a new variable λ' for which $\lambda = \hbar\lambda'$ (notice λ' needs to have appropriate units to compensate for this)

$$x^\mu(\lambda') = \begin{pmatrix} \hbar\omega\lambda' + t_s \\ \hbar k_1\lambda' + x_s \\ \hbar k_2\lambda' + y_s \\ \hbar k_3\lambda' + z_s \end{pmatrix}. \tag{44}$$

This trajectory is identical to the previous one (which you can check by investigating dx/dt for example) but now we have

$$p^\mu = \frac{dx^\mu}{d\lambda'}. \tag{45}$$

This means that we can choose to parameterize a null trajectory in such a way that its four velocity is equal to its four momentum. This is often a convenient parameterization, although the previous one is also natural.