Ramblings on Generalised Functions

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These notes give an explicit example of how to work with generalised functions. In particular, I show how a one-sided Fourier integral may be expressed as the sum of a delta function and the Cauchy Principle value. Note that I have used somewhat offbeat notation in an attempt at clarity.

I. A TALE OF TWO INTEGRALS

My friend Eric has got himself into formal trouble.... Formally following the usual rules for integrals, he's ended up with an expression containing

$$\int_0^\infty e^{ikx} dk. \tag{1}$$

It mightn't be so bad, except for the fact that this is *physics*, and we expect some *physical* answer. Not some damn non-convergent integral!

In any case, the hunt is on for the culprit. It turned out that working backward to the origin of the expression, we found something like the following:

$$\int_0^\infty \left(\int_{-\infty}^\infty f(x) e^{ikx} dx \right) dk \tag{2}$$

where, crucially, $f(x)$ is quite a nice function — something smooth that converges respectfully quickly to zero at large x. Using the usual rules, the natural thing to do with this expression is to change the order of integrals and integrate out the k dependence:

$$\int_0^\infty \left(\int_{-\infty}^\infty f(x) e^{ikx} dx \right) dk \neq \int_{-\infty}^\infty f(x) \left(\int_0^\infty e^{ikx} dk \right) dx.$$

However, as indicated by the $\neq$ symbol, this is incorrect - the function $f(x)e^{ikx}$ needs to be absolutely integrable (Fubini's theorem) before we may change the order of the integrals.

II. GENERALISED FUNCTIONS TO THE RESCUE

The next question we might ask is the following: Is there some sort of useful placeholder which we can use to indicate the action of integrating from 0 to ∞ in the manner above, eliminating all reference to k? The answer is of course that such a placeholder exists, but the challenge is in the word "useful" — here taken to mean "able to be manipulated to make the task of evaluating (2) easier".

I turns out that we may view the action of (2) as a *generalised function*. Under one common formalisation, this means a linear functional — a function on functions which takes a function and gives back a number. It's possible to write such generalised functions in many different forms; one possible transformation is the following:

$$
\begin{aligned}
\int_0^\infty \left(\int_{-\infty}^\infty f(x) e^{ikx} dx \right) dk &= \int_0^\infty \lim_{\epsilon \to 0} e^{-\epsilon k} \left(\int_{-\infty}^\infty f(x) e^{ixk} dx \right) dk \\
&= \int_0^\infty \lim_{\epsilon \to 0} \left(\int_{-\infty}^\infty f(x) e^{(ix-\epsilon)k} dx \right) dk \\
&= \lim_{\epsilon \to 0} \int_0^\infty \left(\int_{-\infty}^\infty f(x) e^{(ix-\epsilon)k} dx \right) dk \\
&= \lim_{\epsilon \to 0} \int_{-\infty}^\infty \left(\int_0^\infty f(x) e^{(ix-\epsilon)k} dk \right) dx
\end{aligned}
$$

where changing the order of integrals is now allowed because the modified integrand with finite ϵ is absolutely convergent. Continuing, we may pull out the factor of $f(x)$ and perform the integral over k:

$$
\begin{aligned}
&= \lim_{\epsilon \to 0} \int_{-\infty}^\infty f(x) \left[\frac{1}{ix - \epsilon} e^{(ix-\epsilon)k} \right]_0^\infty dx \\
&= \lim_{\epsilon \to 0} \int_{-\infty}^\infty f(x) \frac{1}{\epsilon - ix} dx
\end{aligned}
$$

We end up with the interesting result that

$$
\int_0^\infty \bullet\, e^{ikx} dk = \lim_{\epsilon \to 0} \bullet\, \frac{1}{\epsilon - ix} \tag{3}
$$

where the symbol $\bullet$ is used to indicate that an integral with respect to x over some nice function $f(x)$ must be inserted for the expression to make sense. Of course, the natural shorthand for such expressions is to omit this extra piece of notation and simply write

$$
\int_0^\infty e^{ikx} dk = \lim_{\epsilon \to 0} \frac{1}{\epsilon - ix}.
$$

Unfortunately, this kind of expression leads to much confusion among poor unsuspecting physics students — how may a clearly non-convergent integral be equal to a limit which equally clearly exists and evaluates to $-1/ix$?

Note that I've been a little blithe about changing the order of integrals and limits in the third line of the derivation above. It's fairly clear from inspecting the result that we need to be careful when doing this. However, I believe that imposing suitable niceness conditions such as

$$
\int_{-\infty}^\infty \left| \int_0^\infty f(x) e^{ikx} dx \right| dk < \infty
$$

on the "half Fourier transform of f" allows us to use the dominated convergence theorem of Lebesgue integration, and everything is OK. This may be an unnecessarily strong condition, but I'm going to gloss over the detailed analysis for the moment.

III. EXPRESSING THE PROBLEM IN TERMS OF "MORE FAMILIAR" GENERALISED FUNCTIONS

It turns out that generalised functions *do* allow the integral (2) to be transformed into other useful forms. One possible transformation is performed in [1] — we flesh out their procedure in the following.

A. Representations of the Dirac delta function

Firstly, we note the definition of the Dirac delta function in several notations:

- As a linear functional:
$$\delta(f) := f(0)$$
- In the usual physics notation:
$$\int_{-\infty}^{\infty} \delta(x) f(x) dx := f(0)$$
- In my notation from above:
$$\bullet\, \delta(x) := f(0)$$

Delta functions occur naturally in theories which make use of continuous basis transformations. The prime example is the theory of Fourier transformations in which the Fourier transform of a function f may be defined by

$$F(k) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx.$$

This transformation has an inverse of the same form:

$$f(x') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{ikx'} dk,$$

and putting these two expressions together leads us directly to the delta function. We have

$$\begin{aligned} f(x') &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x) e^{-ikx} dx \right) e^{ikx'} dk \\ \implies f(0) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x) e^{-ikx} dx \right) dk \end{aligned}$$

or in the dot notation,

$$\bullet\, \delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bullet\, e^{ikx} dk.$$

(Note that the negative sign in e^{-ikx} has been removed using the symmetry in k.)

Starting from the representation of δ above, we proceed to obtain a representation of δ which will allow us to relate the original integral (2) to a combination of the delta function and the Cauchy principle value. We have

$$\begin{aligned}\bullet\,\delta(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \bullet\, e^{ikx}\, dk \\ &= \frac{1}{2\pi}\left(\int_{-\infty}^{0} \bullet\, e^{ikx}\, dk + \int_{0}^{\infty} \bullet\, e^{ikx}\, dk\right) \\ &= \frac{1}{2\pi}\left(\int_{0}^{\infty} \bullet\, e^{-ikx}\, dk + \int_{0}^{\infty} \bullet\, e^{ikx}\, dk\right).\end{aligned}$$

Finally, using equation (3) yields

$$\begin{aligned}\bullet\,\delta(x) &= \frac{1}{2\pi}\left(\lim_{\epsilon\to 0}\bullet\,\frac{1}{\epsilon+ix} + \lim_{\epsilon\to 0}\bullet\,\frac{1}{\epsilon-ix}\right) \\ &= \frac{1}{2\pi}\left(\lim_{\epsilon\to 0}\bullet\,\left(\frac{1}{\epsilon+ix}+\frac{1}{\epsilon-ix}\right)\right) \qquad (4)\end{aligned}$$

B. The Cauchy principle value

The Cauchy principle of an integral may be used to define a less familiar generalised function as follows. First, we define the principle value of the integral of a function, f:

$$P(f) \equiv P\int_{-\infty}^{\infty} f(x)dx := \lim_{\epsilon\to 0}\int_{\epsilon<|x|} f(x)dx = \lim_{\epsilon\to 0}\left(\int_{-\infty}^{-\epsilon} f(x)dx + \int_{\epsilon}^{\infty} f(x)dx\right).$$

This defines one possible method for obtaining a number from the integral of a function, f which has a singularity at the origin. It turns out that the principle value of $\int f(x)/x\, dx$ is often a useful quantity, and will help us in manipulating equation (3).

With a little effort, it is possible to show [2] that an alternative expression for $P(f/x)$ is

$$P(f/x) = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} f(x)\frac{x}{x^2+\epsilon^2}dx.$$

Adopting the notation

$$\int_{-\infty}^{\infty} f(x)\frac{P}{x}dx := P(f/x),$$

we may write

$$\bullet\,\frac{P}{x} = \lim_{\epsilon\to 0}\bullet\,\frac{x}{x^2+\epsilon^2}.$$

Rearranging this equation leads to a form which we can use below:

$$\bullet\,\frac{P}{x} = \frac{1}{2i}\lim_{\epsilon\to 0}\bullet\,\left(\frac{1}{\epsilon-ix} - \frac{1}{\epsilon+ix}\right). \qquad (5)$$

C. Putting it all together — an expression for the half-Fourier transform

Combining equations (5) and (4), we have

$$\lim_{\epsilon\to 0} \bullet \frac{1}{\epsilon \pm ix} = \pi \bullet \delta(x) \mp i \bullet \frac{P}{x}.$$

This finally allows us to express equation (2) in terms of the principle value and the delta function:

$$\begin{aligned}\int_0^\infty \left(\int_{-\infty}^\infty f(x) e^{ikx} dx \right) dk &= \lim_{\epsilon\to 0} \int_{-\infty}^\infty f(x) \frac{1}{\epsilon - ix} dx \\ &= \int_{-\infty}^\infty f(x) \left(\pi\delta(x) + i\frac{P}{x} \right) dx \\ &= \pi f(0) + iP(f/x).\end{aligned}$$

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[1] Michael Fowler, *Fourier Series, Fourier Transforms and the Delta Function*, http://galileo.phys.virginia.edu/classes/751.mf1i.fall02/FourierSeries.htm, 2004; Last accessed 9/2/2006.

[2] Proving this amounts to proving that

$$\lim_{\epsilon\to 0} \left(\int_\epsilon^\infty \frac{f(x)}{x} dx - \int_0^\infty \frac{x}{x^2+\epsilon^2} dx \right) = \lim_{\epsilon\to 0} \epsilon^2 \int_\epsilon^\infty \frac{1}{x(x^2+\epsilon^2)} dx,$$

which may be used to show

$$\lim_{\epsilon\to 0} \left(\int_{-\infty}^\infty \frac{x}{x^2+\epsilon^2} f(x) dx - \int_{\epsilon<|x|} \frac{f(x)}{x} dx \right) = 0.$$

This does the trick after rearranging and using the definition for $P(f/x)$.