Entanglement and boundary critical phenomena

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Reference: H.-Q. Zhou et al., Phys. Rev. A 74, 050305(R) (2006)
(cond-mat/0511732)

Outline

Entanglement and quantum criticality in 1D systems

- Review of some previous work: Bulk systems
- Our work: Systems with boundaries

Quantum phase transitions

Consider a Hamiltonian which depends on parameter λ: $H = H(\lambda)$

Suppose that there is a quantum phase transition at $\lambda = \lambda_c$

- Qualitative difference in ground state for $\lambda < \lambda_c$ and $\lambda > \lambda_c$
- Typically, at the transition point $\lambda = \lambda_c$, the ground state energy $E_0(\lambda)$ is non-analytic

For a continuous (aka “critical”, “second-order”) phase transition:

As $\lambda \to \lambda_c$: - characteristic length scale (correlation length) $\xi \to \infty$
- associated characteristic energy scale (gap) $\Delta \to 0$

$\lambda = \lambda_c$ is a quantum critical point:

- Scale invariant ($\xi = \infty$)
- Gapless spectrum ($\Delta = 0$)

Example: S = 1/2 Ising chain in a transverse magnetic field

$$H = -\sum_j \left(S_j^z S_{j+1}^z + \lambda S_j^x \right)$$

$\lambda \neq 1/2$: spectrum gapped

$\lambda = 1/2$: spectrum gapless

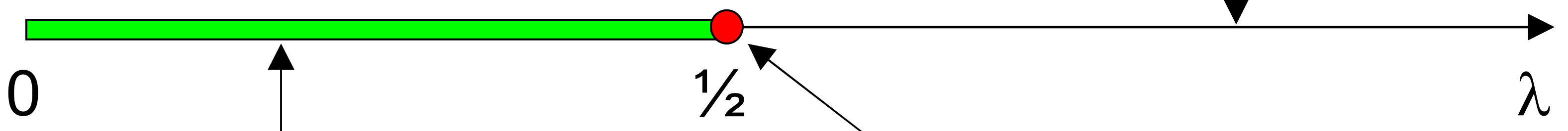
"Connected" correlation function:

$$G_{i,j}^c = \langle S_i^z S_j^z \rangle - \langle S_i^z \rangle \langle S_j^z \rangle$$

$\lambda > 1/2$: Paramagnet

$$G_{i,j}^c \sim e^{-|x_i - x_j|/\xi}$$

Zero-temperature phase diagram:



$0 \leq \lambda < 1/2$: Ferromagnet

$$G_{i,j}^c \sim e^{-|x_i - x_j|/\xi}$$

$\lambda = 1/2$: Quantum critical point

$$G_{i,j}^c \sim |x_i - x_j|^{-1/4}$$

The low-energy, long-distance physics of this and many other quantum critical (i.e., gapless) systems in one spatial dimension is described by a conformal field theory (CFT)

A CFT is characterized by a number c, the “central charge”, which labels a universality class

Example: Ising chain in a transverse field at criticality:
CFT with $c = 1/2$

Entanglement and quantum criticality

Entanglement = quantum correlations

As a system is tuned across a quantum critical point, the ground state and its correlations undergo qualitative changes.

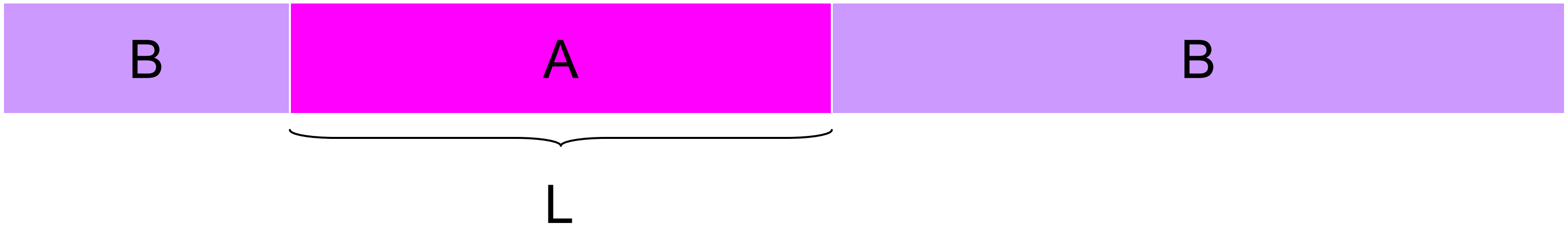
- Is the same true of the entanglement in the ground state?
  Can entanglement be used to characterize/classify phase transitions?
- How does one best quantify entanglement in a many-body system?

Early influential work:
A. Osterloh et al., Nature 416, 608 (2002)
T. J. Osborne and M. A. Nielsen, Phys. Rev. A 66, 032110 (2002)

Entanglement and quantum criticality

G. Vidal et al., Phys. Rev. Lett. 90, 227902 (2003)

Considered the ground-state von Neumann entropy between a block of length L and the rest of the spin chain



Reduced density matrix for A: $\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi|$

von Neumann (entanglement) entropy: $S = -\mathrm{Tr}_A \rho_A \log \rho_A$

At criticality: $S \approx \frac{c}{3} \log(L/a) + \mathrm{const.}$ $(a = \text{lattice constant})$

i.e. diverges logarithmically with L with universal prefactor (*c* = central charge!)

Away from criticality: $S \approx \frac{c}{3} \log(\xi/a) + \mathrm{const.}$

for L $\gtrsim \xi$ (the correlation length), i.e. saturates for large L

Entanglement (von Neumann) entropy and conformal field theory (CFT)

Critical chain of infinite size, block length L:

$$\boxed{S = \frac{c}{3}\log\frac{L}{a} + k}$$

c : central charge of CFT (universal)
k : non-universal constant

Proof from CFT [Calabrese and Cardy, J. Stat. Mech. P06002 (2004), earlier work by Holzhey, Larsen, Wilczek, Nucl. Phys. B 424, 44 (1994)]

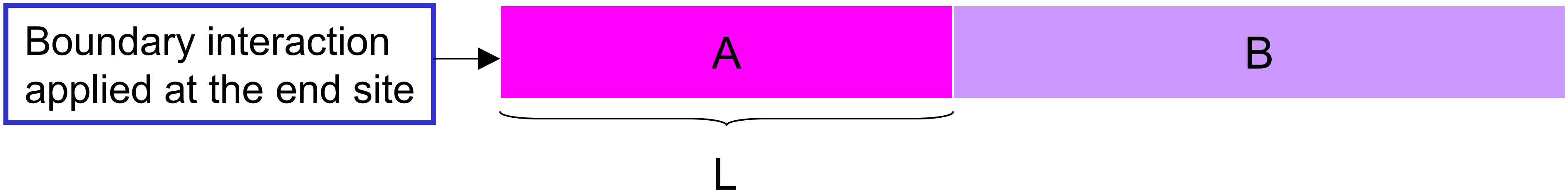
Systems with boundaries

Relevant to a diverse range of problems, such as:

- Quantum spin chains with boundaries
- Impurities in Luttinger liquids
- Kondo physics
- Tunneling in fractional quantum Hall devices
- Open string theory

Entanglement (von Neumann) entropy for spin chains with a boundary

Semi-infinite critical spin chain with block (A) of length L



$$\boxed{S = \frac{c}{6}\log\frac{2L}{a} + \frac{k}{2} + S_b}$$

Derived from (boundary) CFT:
Calabrese and Cardy (CC), 2004
Zhou et al., PRA 74, 050305(R) (2006)
(corrected factors of 2 in CC paper)

k: same non-universal constant
as for infinite chain

$$\boxed{S_b = \log g \text{ : Boundary entropy}}$$

Affleck & Ludwig, PRL 67, 161 (1991)

$$g = \langle B|0\rangle$$

|B⟩: ”boundary state”
|0⟩: ground state

Cardy, Nucl. Phys. B 324, 581 (1989)

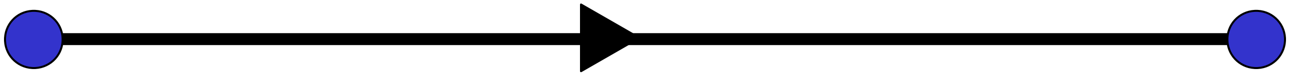
Semi-infinite S = 1/2 transverse Ising chain with a boundary magnetic field

$$H = -\sum_{j=0}^{\infty} \left(S_j^z S_{j+1}^z + \lambda S_j^x \right) + h_b S_0^z$$

boundary magnetic field

Set λ = 1/2 so that model
is bulk critical with c = 1/2

Renormalization group (RG) flow of boundary magnetic field:

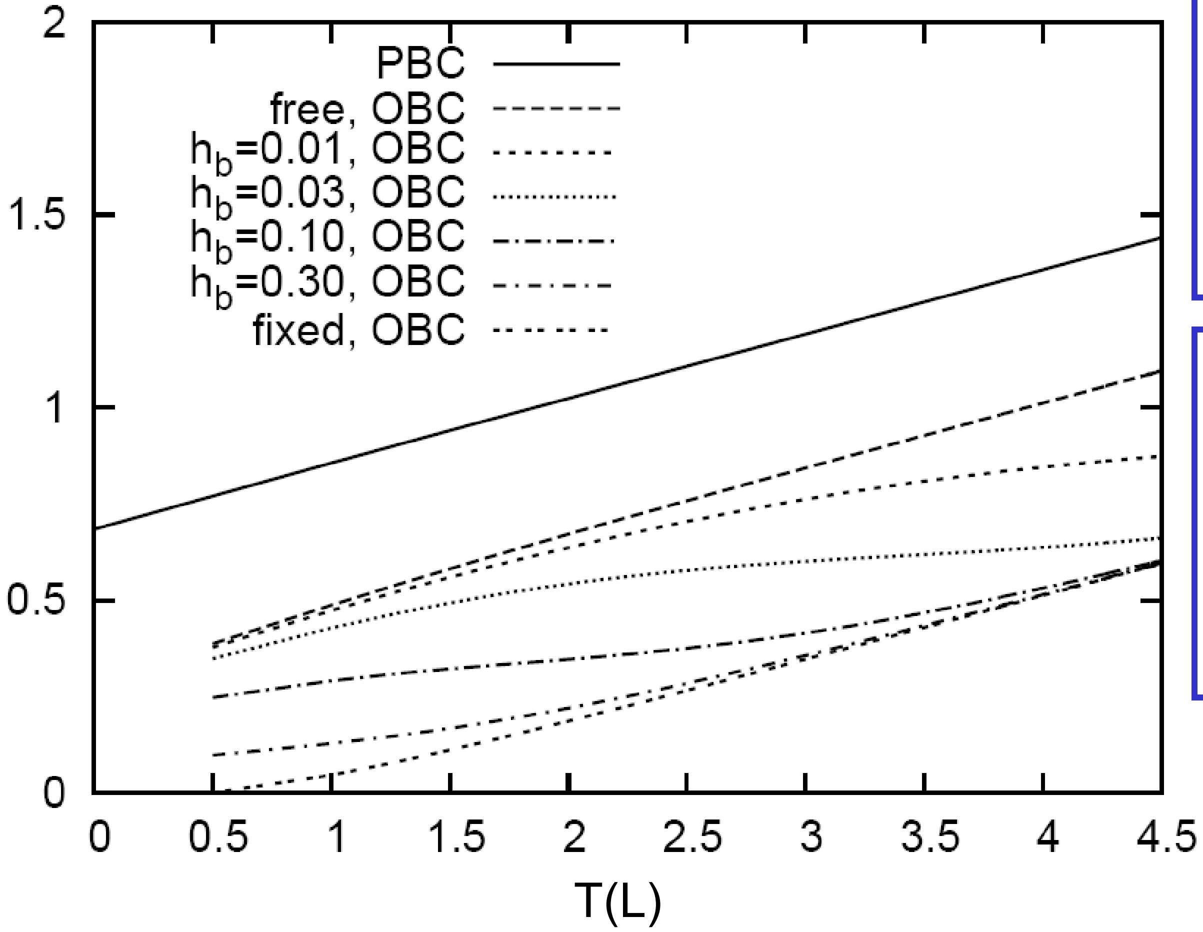


$h_b = 0$ $\quad g = 1$

$h_b = \infty$ $\quad g = 1/\sqrt{2}$

g-theorem: “g decreases along boundary RG flows”
[Affleck & Ludwig, PRL 67, 161 (1991)]

Extracting boundary entropies from DMRG calculations


von Neumann entropy

PBC:

$$S_{\mathrm{PBC}} = \frac{c}{3}\, T_{\mathrm{PBC}}(L) + k$$

$$T_{\mathrm{PBC}}(L) = \log\left(\frac{N}{\pi a} \sin \frac{\pi L}{N}\right)$$

N = 80

OBC:

$$S_{\mathrm{OBC}} = \frac{c}{3}\, T_{\mathrm{OBC}}(L) + \frac{k}{2} + S_b$$

$$T_{\mathrm{OBC}}(L) = \frac{1}{2} \log\left(\frac{2N}{\pi a} \sin \frac{\pi L}{N}\right)$$

N = 800

(for OBC: same BC's at both ends)

Fits give:

Free BC (h_b = 0): S_b = -0.003 ≈ log 1

Fixed BC (h_b = ∞): S_b = -0.497 ≈ log (1/√2)

Curves for nonzero h_b cross over to curve for h_b = ∞ for large enough L (crossover length decreases with increasing h_b)

Majorization

Previous work: Entanglement loss along bulk RG flows can be given a more “fine-grained” characterization in terms of majorization:
Latorre et al., PRA 71, 034301 (2005)
Orus, PRA 71, 052327 (2005)

Consider two probability distributions:

$$\lambda = \{\lambda_i\}, \quad \mu = \{\mu_i\}$$

Order the elements such that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$$
$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r$$

r = largest number of nonzero elements in λ or μ

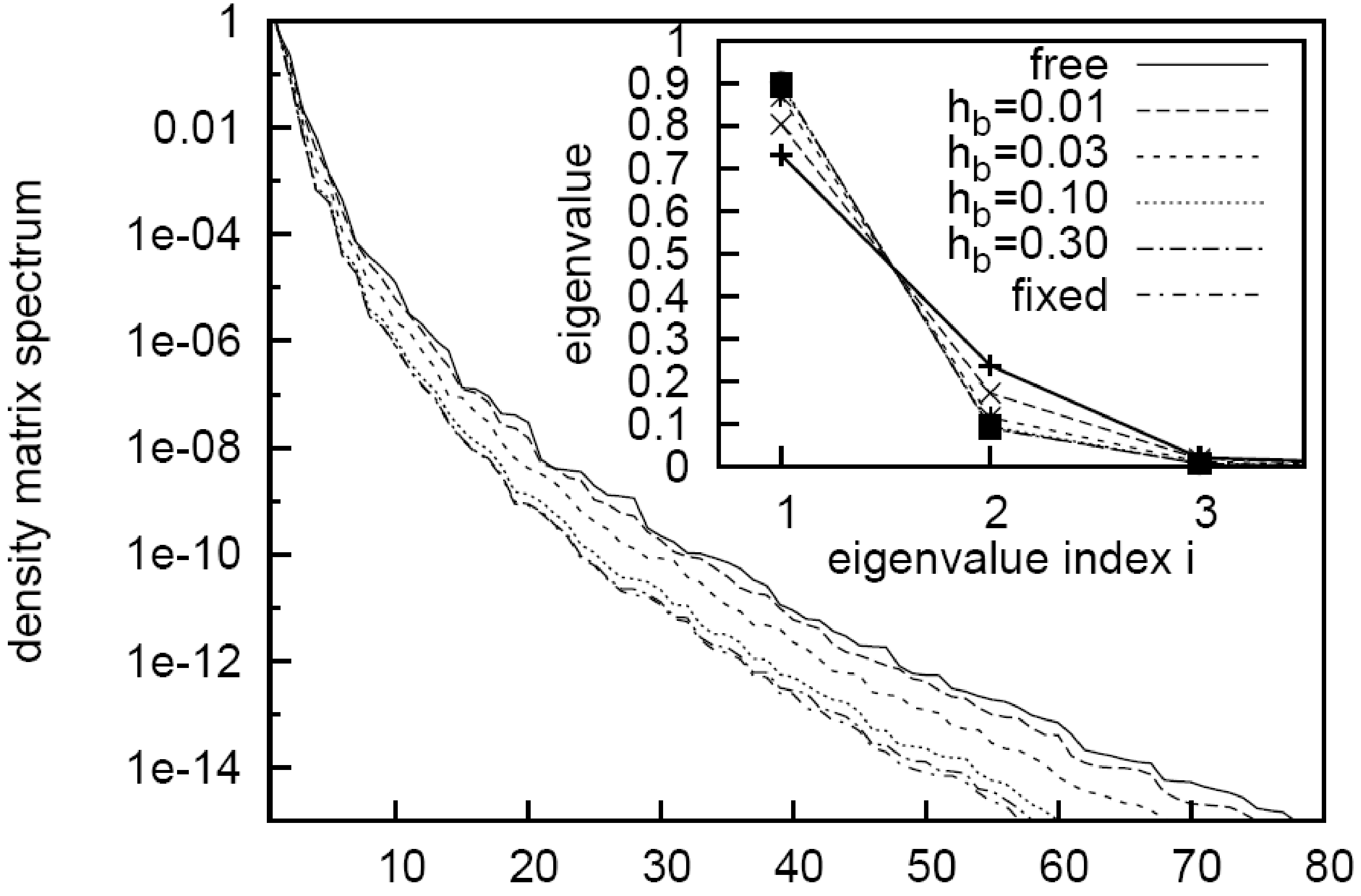
If

$$\sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i, \quad k = 1, \ldots, r$$

then λ is majorized by μ (roughly: λ is “more disordered” than μ)

In our problem: probability distributions λ and μ formed by eigenvalues of the reduced density matrix of block for two different values of h_b, and r is the Schmidt rank

Majorization along boundary RG flows



L = 128

- Eigenvalue distributions cross only once ⇒ majorization
- Can also be argued from boundary conformal field theory

Conclusions

- Found CFT prediction for entanglement (von Neumann) entropy for spin chains with boundaries
  (corrected factors of 2 in Calabrese-Cardy paper)

- Extracted boundary entropies numerically by fitting DMRG calculations for entanglement entropy to the CFT predictions

- Demonstrated majorization along boundary RG flows
  (CFT prediction and numerical DMRG calculations)

- For more details about these and some other topics not discussed here (Renyi entropy, single-copy entanglement, S=1/2 XXZ chain with boundary magnetic field etc.), see:

H.-Q. Zhou et al., Phys. Rev. A 74, 050305(R) (2006) (cond-mat/0511732)