Asymptotic solutions to the Gross-Pitaevskii gain equation: Growth of a Bose-Einstein condensate

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We give an asymptotic analytic solution for the generic atom-laser system with gain in a D-dimensional trap, and show that this has a non-Thomas-Fermi behavior. The effect is due to Bose-enhanced condensate growth, which creates a local-density maximum and a corresponding outward momentum component. In addition, the solution predicts amplified center-of-mass oscillations, leading to enhanced center-of-mass temperature.

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I. INTRODUCTION

The description of Bose-Einstein condensate (BEC) growth [1–4] has become important due to the need to understand the physics of atom lasers [5]. These recently developed devices that emit coherent wavelike beams of atoms promise a new generation of precision measurements, applications in nanotechnology, and novel tests of fundamental concepts in quantum theory. While it is possible to perform first-principles numerical simulations of the relevant equations [6], a great deal of physical insight can be obtained from an analytic solution.

In this paper, we give an analytic asymptotic solution to the Gross-Pitaevskii equation describing the early stages of condensate growth in a trap. The physical insight we obtain from this is that a growing nonequilibrium condensate has a nonuniform momentum distribution across the condensed region. As a result, the observed density behaves as though the trap frequency is increased, relative to the usual Thomas-Fermi behavior [7] of an equilibrium BEC. In addition, our solutions show center-of-mass oscillations whose kinetic energy is amplified with the condensate growth. This implies that, while BEC's formed through evaporative cooling may have a low temperature for their internal degrees of freedom, the temperature for the center-of-mass motion is higher, leading to increased noise in the velocity and direction of atom laser beams.

II. GROSS-PITAEVSKII GAIN MODEL

We start by considering a commonly used model of a one-component trapped Bose-Einstein condensate—the Gross-Pitaevskii (GP) equation [8] modified by a linear gain term g [9], of the form

$$\frac{\partial \Psi(\mathbf{x},t)}{\partial t}=\left[g-i\left(\frac{\hbar}{2m}\boldsymbol{\nabla}^2+V(\mathbf{x})+U|\Psi|^2\right)\right]\Psi. \tag{1}$$

Here $\Psi(\mathbf{x},t)$ is the mean-field amplitude [so that $|\Psi(\mathbf{x},t)|^2$ is the particle number density], m is the atomic mass, and U is the effective interaction potential. In the treatment of D equal to one, two, or three space dimensions, U is given by $U=4\pi\hbar a L^{3-D}/m$, where a is the scattering length and L is the confinement length. The potential term $V(\mathbf{x})$ is due to an optical or magnetic trap, which we assume is harmonic. In the simplest rotationally symmetric case, the trap potential is given by $V(\mathbf{x})=m\omega^2|\mathbf{x}|^2/(2\hbar)$, where ω is the trap oscillation frequency.

It is helpful to understand the physical implications of Eq. (1) by an analysis of the relevant time scales. In the present case, there are three different relevant time scales. These are the time scale for the condensate growth, $t_g=1/g$; the trap oscillation time scale, $t_\omega=1/\omega$; and the ''healing'' time scale, $t_h=1/[U|\Psi(\mathbf{x}_0,t)|^2]$, associated with the mean-field interaction potential at the condensate center of mass $\mathbf{x}_0$.

Since current BEC's typically have a relatively high density, the healing time is usually the smallest, which results in a Thomas-Fermi (TF) type equilibrium condensate. If the gain time scale is long enough, then one may expect that even during growth, the condensate will adiabatically follow the TF solution. This is commonly assumed in analyzing experimental data [1]. Another possibility is that the gain time scale is shorter than the trap oscillation period, in which case we should no longer expect adiabatic TF-like behavior.

A. Asymptotically growing solution

To show this distinction, we now proceed to give an analytic nonequilibrium solution of the GP equation with gain. First, we expand the field in terms of the amplitude and phase,

$$\Psi(\mathbf{x},t)=A(\mathbf{x},t)e^{-i\phi(\mathbf{x},t)}/\sqrt{U}, \tag{2}$$

and obtain the following coupled equations:

$$\frac{\partial A}{\partial t}=gA+\frac{\hbar}{2m}A\boldsymbol{\nabla}^2\phi+\frac{\hbar}{m}\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}A, \tag{3}$$

$$\frac{\partial \phi}{\partial t}=A^2+\frac{m\omega^2}{2\hbar}|\mathbf{x}|^2+\frac{\hbar}{2m}\left(|\boldsymbol{\nabla}\phi|^2-\frac{1}{A}\boldsymbol{\nabla}^2A\right). \tag{4}$$

Next, we wish to investigate possible asymptotic solutions for long times, i.e., steadily growing solutions, valid some time after initial nucleation of the condensate, yet before any gain saturation has occurred. We consider first the rotationally symmetric case, where $\boldsymbol{\nabla}^2=\partial_r^2+[(D-1)/r]\partial_r$, with $r=|\mathbf{x}|$. Following the construction successfully used recently in optical fiber environments [10]—and extending these to the current multidimensional case of a trapped BEC—we suppose that the amplitude has a self-similar behavior at large time. The phase is assumed to depend on the atomic density at the condensate center, and to have a uniform chirp giving a radially dependent outward momentum. Thus, we seek a solution in the form of

$$A=\lambda(t)f(r/\lambda(t)), \tag{5}$$

$$\phi=\frac{\lambda^2(t)}{2\tilde{g}\tau}-\frac{m\tilde{g}}{2\hbar}r^2. \tag{6}$$

Here $\lambda(t)$ is a scaling function, while τ and $\tilde{g}$ are unknown coefficients. Using this construction, we find, from Eqs. (3) and (4),

$$\frac{\partial\lambda(t)}{\partial t}\left(f-\frac{rf'}{\lambda}\right)=g\lambda f-f'\tilde{g}r-\frac{D}{2}\lambda f\tilde{g}, \tag{7}$$

$$\frac{\partial\lambda(t)}{\partial t}\left(\frac{1}{\tilde{g}\tau}\lambda\right)=(\lambda f)^2+\frac{\hbar\tilde{\omega}^2}{2m}r^2-\frac{\hbar}{2m}\left(\frac{f''}{f\lambda^2}+\frac{D-1}{f\lambda r}f'\right), \tag{8}$$

where we have defined $\tilde{\omega}^2\equiv\omega^2+\tilde{g}^2$.

From the first (amplitude) equation, the solution requires the twin conditions that

$$\frac{\partial\lambda(t)}{\partial t}=\left(g-\frac{D\tilde{g}}{2}\right)\lambda=\tilde{g}\lambda. \tag{9}$$

This implies that the scaling function λ grows exponentially with time, having a solution of $\lambda(t)=\exp(\tilde{g}t)$, where we can immediately solve for the growth rate, since clearly $\tilde{g}=2g/(D+2)$. The physical interpretation of this equation is rather straightforward; the growth of the amplitude at any radial point is reduced below the single-mode growth rate g, due to an outward flow of atoms, which transfers part of the increased density to a larger radius. This effect increases with dimensionality of the space.

The phase equation can now be simplified using the fact that the last two terms involving derivatives have terms in $1/\lambda$, and hence become exponentially smaller than the earlier terms, at long times, as the healing time becomes smaller. This immediately gives the following solution for f:

$$f=\begin{cases}\sqrt{[1-|\mathbf{x}|^2/R(t)^2]/\tau} & \text{for } |\mathbf{x}|<R(t),\\ 0 & \text{for } |\mathbf{x}|>R(t),\end{cases} \tag{10}$$

where $R(t)$ is the maximum radius given by

$$R(t)=e^{\tilde{g}t}\sqrt{2\hbar/(m\tau\tilde{\omega}^2)}. \tag{11}$$

The remaining unknown constant τ is obtained by evaluating the integral $N(t)=\int|\Psi(\mathbf{x},t)|^2d^D\mathbf{x}$ for the total number of particles:

$$N(t)=e^{\tilde{g}(D+2)t}\left(\frac{2\hbar}{m\tilde{\omega}^2}\right)^{D/2}\frac{2C_D\,\tau^{-(D+2)/2}}{UD(D+2)}. \tag{12}$$

Here C_D is a dimension-dependent constant, with $C_1=2, C_2=2\pi$, and $C_3=4\pi$. This gives the expected result that the overall atom number grows exponentially with a gain constant of $2g=\tilde{g}(D+2)$; that is, $N(t)=N(0)\exp(2gt)$. The reason for this apparent difference in gain constants is that the relatively slower growth in density at the center of the condensate is exactly compensated for by the increase in condensate radius with time.

The constant τ is solved in terms of $N(0)$, so that

$$\tau=\left[\left(\frac{2\hbar}{m}\right)^{D/2}\frac{2C_D}{UD(D+2)N(0)\tilde{\omega}^D}\right]^{2/(D+2)}. \tag{13}$$

This coincides with the healing time for a condensate with atom number $N(0)$, in TF equilibrium, but with an effective trap oscillation frequency $\tilde{\omega}$ instead of ω.

The final result for the particle number density $|\Psi(\mathbf{x},t)|^2=\lambda^2f^2/U$ is

$$|\Psi(\mathbf{x},t)|^2=\frac{e^{2\tilde{g}t}}{\tau U}\left(1-\frac{|\mathbf{x}|^2}{R(t)^2}\right). \tag{14}$$

While this gives an asymptotic solution in terms of the initial atom number, the solution needs to be compared with the usual TF solution to understand the physical implications. In particular, we can notice an interesting scaling behavior for the radius, as a function of the trap frequency. In the present case, the radius scales as

$$R\propto\tilde{\omega}^{-2/(D+2)}=(\omega^2+\tilde{g}^2)^{-1/(D+2)}. \tag{15}$$

In equilibrium TF solutions, there is a similar behavior, except that there is no outward momentum term. As a consequence, the TF radius ($R_{\mathrm{TF}}\propto\omega^{-2/(D+2)}$) is always larger at a given total atom number, and the density is lower at the center, than in these nonequilibrium solutions with gain present. The physical reason for this is simply the rapid Bose-enhanced increase in number density (and hence pressure) at the condensate center during the nucleation process. This effect becomes appreciable when $g\gg\omega$ or $t_g\ll 1/\omega$, so that the time scale for condensate growth is faster than the trap oscillation period.

B. Numerical results

We now turn to comparisons between the analytic asymptotic solution and exact numerical results. In Fig. 1, we plot the rms radius $\overline{r(t)}$ of a growing BEC versus t. The result found from the above asymptotic solution in three dimensions is simply given by $\overline{r(t)}=\sqrt{3/7}R(t)$. This is represented by the dashed line, for $g=10\omega$ and $N(0)=10$. For comparison, the TF rms radius for the same values of $N(t)$ is represented by the dotted line. The full lines correspond to the results of direct numerical simulations of the GP equation with gain, with initial Gaussian wave functions of different widths. The values of other parameters are chosen to correspond to a ^{87}Rb BEC ($m=1.44$ kg, $U_3=5\times10^{-17}$ m^3/s) in a trap with $\omega/2\pi=100$ Hz. As one can see from the graph,

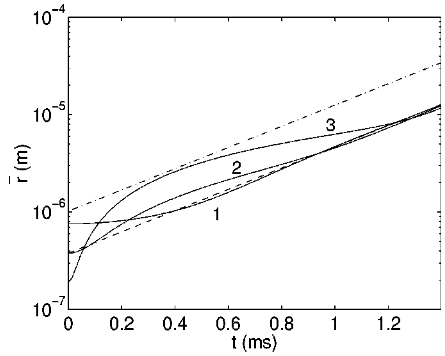

FIG. 1. The rms radius $\overline{r}$ (in a logarithmic scale) of a growing condensate versus t, for $g=10\omega$. The full lines 1, 2, and 3 correspond, respectively, to initial Gaussian wave functions having rms radii that are twice as larger as, equal to, and twice as small as the asymptotic rms radius $\overline{r}(0)=\sqrt{3/7}R(0)$. The dashed line is the asymptotic result, while the dotted line corresponds to the equilibrium (TF) solution.

despite the initial differences, the mean radii approach the same asymptotic value of $\sim\sqrt{3/7}R(t)$, which is different from the TF result for the same total number of particles.

The outward momentum density $p=|\mathbf{p}|$ (where $\mathbf{p}=-i\hbar[\Psi^*(\nabla\Psi)-\text{H.c.}]/2$) is plotted in Fig. 2, showing that in a growing BEC the flow of atoms from the trap center increases initially with the distance, reaches a maximum, and vanishes at $r\gtrsim R$.

III. ASYMMETRIC CASE AND CENTER-OF-MASS OSCILLATIONS

The above rotationally symmetric results can easily be generalized to asymmetric cases, where the trap oscillation frequencies ω_i in each space direction are different. In this case, the trap potential term in Eqs. (1) and (4) is replaced by

$$V(\mathbf{x})=\frac{m}{2\hbar}\sum_{i=1}^{D}\omega_i^2x_i^2. \tag{16}$$

Assuming again that the amplitude has a self-similar behavior at long times, one can use the previous construction. In addition, we find that a more general solution can be obtained. Following the approach of Kohn's theorem [11], generalized to include gain, we allow for center-of-mass oscillation of the growing condensate, independent of the interparticle interactions.

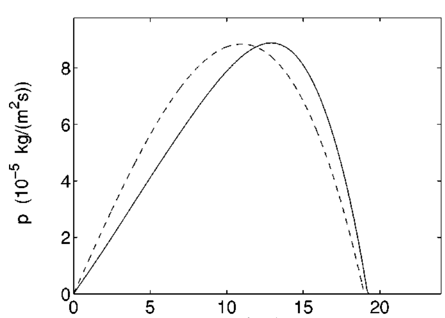

FIG. 2. The momentum density p versus the radial distance from the center of the trap r. The full line corresponds to the resulting distribution of case 1 in Fig. 1; the dashed line is the corresponding asymptotic solution.

Thus, we seek a solution in the form of

$$A=\lambda(t)f[\Delta\mathbf{x}(t)/\lambda(t)],$$

$$\phi=\phi_0(t)+\frac{\lambda(t)^2}{2\tilde{g}\tau}-\frac{m}{\hbar}\left[\frac{\tilde{g}\Delta\mathbf{x}(t)^2}{2}+\Delta\mathbf{x}(t)\cdot\dot{\mathbf{x}}_0(t)\right],$$

where $\Delta\mathbf{x}(t)\equiv\mathbf{x}-\mathbf{x}_0(t)$, $\mathbf{x}_0(t)$ is the center of mass, and the dot stands for the time derivative.

Using the same procedure as before, we first find—from the amplitude equation—that the function $\lambda(t)$ is given by

$$\lambda(t)=\exp(\tilde{g}t), \tag{17}$$

where $\tilde{g}=2g/(D+2)$, as previously.

To treat the phase equation, in which we neglect the last term $\propto 1/\lambda$ that becomes exponentially small at long times, we assume that

$$f^2=F_1+F_2(\Delta\mathbf{x})^2,$$

where the functions F_1 and F_2 are to be found by equating the terms in powers of Δx_i. From the terms in Δx_i, we find that each component of $\mathbf{x}_0(t)$ satisfies the equation $\ddot{x}_{0,i}+\omega_i x_{0,i}=0$, so that the condensate center of mass oscillates according to

$$x_{0,i}(t)=a_{0,i}\cos(\omega_i t+\delta_{0,i}), \tag{18}$$

where $a_{0,i}$ and $\delta_{0,i}$ are the initial amplitude and phase.

By equating the terms in $(\Delta x_i)^2$ and the zeroth-order terms, respectively, we obtain an equation for $\phi_0(t)$,

$$\dot{\phi}_0+\frac{m}{2h}\sum_{i=1}^{D}[(\dot{x}_{0,i})^2-\omega_i^2x_{0,i}]=0,$$

as well as solutions to F_1 and F_2:

$$F_1=1/\tau,$$

$$F_2=-\frac{m}{2\hbar\lambda(t)}\sum_{i=1}^{D}\frac{(\omega_i^2+\tilde{g}^2)\Delta x_i(t)}{[\Delta\mathbf{x}(t)]^2}.$$

Combining these together and using the solutions for the center-of-mass coordinates, we finally obtain that the solution for the function $\phi_0(t)$ is given by

$$\phi_0(t)=-(m/2\hbar)[\mathbf{x}_0(t)\cdot\partial\dot{\mathbf{x}}_0(t)], \tag{19}$$

while the solution for f is

$$f=\frac{1}{\sqrt{\tau}}\left(1-\sum_{i=1}^{D}\frac{[x_i-x_{0,i}(t)]^2}{R_i(t)^2}\right)^{1/2}, \tag{20}$$

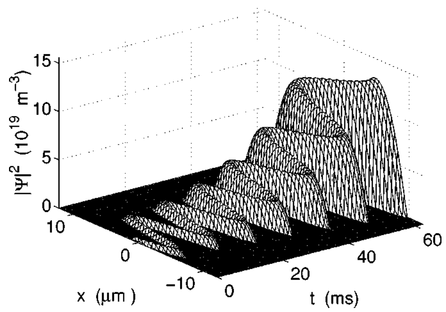

FIG. 3. Growth of a BEC with a center-of-mass oscillation present. Shown is the condensate density $|\Psi(x,t)|^2$, as found from the asymptotic solution in a symmetric trap, with a center-of-mass oscillation in the x direction and $N(0)=10$. The gain coefficient is chosen as $g=0.1\omega$, while $a_{0,1}=4.86$ μm and $\delta_{0,1}=\pi/2$. Other parameter values are as previously.

in the region of space where the expression in large brackets is positive, and $f=0$ outside that region.

The maximum radius $R_i(t)$ in each space direction is

$$R_i(t)=e^{\tilde{g}t}\sqrt{2\hbar/(m\tau\tilde{\omega}_i^2)}, \tag{21}$$

where we have introduced $\tilde{\omega}_i^2\equiv\omega_i^2+\tilde{g}^2$. In addition, the constant τ is solved, as previously, in terms of $N(0)$ by evaluating the integral for the total number of particles $N(t)=N(0)\exp(2gt)$. The resulting expression for τ is given by the same equation as before, Eq. (13), except that $\tilde{\omega}^D$ is replaced by $\Pi_{j=1}^D\tilde{\omega}_j$. This again corresponds to the healing time in a TF condensate, with $N(0)$ particles and effective trap oscillation frequencies $\tilde{\omega}_i$.

The final result for the particle number density is now

$$|\Psi(\mathbf{x},t)|^2=\frac{e^{2\tilde{g}t}}{\tau U}\left(1-\sum_{i=1}^{D}\frac{[x_i-x_{0,i}(t)]^2}{R_i(t)^2}\right). \tag{22}$$

An example showing growth of the BEC, while the condensate center of mass oscillates, is shown in Fig. 3.

IV. SUMMARY

The physical interpretation of these results is that the asymptotic solution with gain has a density distribution that is similar to the TF solution, except that the trap oscillation effective frequency is increased, with $\omega_i^2\rightarrow\tilde{\omega}_i^2=\omega_i^2+\tilde{g}^2$. For a given gain constant, this has the strongest effect for weakly trapped (low-frequency) directions in an asymmetric trap. The effective frequency $\tilde{\omega}_i$ only modifies the density distribution, since the center of mass still oscillates with the original trap oscillation frequency ω_i. In addition, the solution gives an outward momentum component due to the bosonic stimulation effect, which is stronger in the center of the trap.

The results obtained show that Bose stimulation can occur to moving condensates as easily as to stationary ones. In cases such as this, the amplitude of the center-of-mass oscillation does not change during the condensate growth, while the total mass of the condensate increases. This means that the condensate center-of-mass kinetic energy, $E_K=N(t)m\Sigma_i\omega_i^2a_{0,i}^2$, can grow exponentially to large values, even in the absence of technical noise. While the present model of gain is simplified, similar types of center-of-mass motion are found in first-principles simulations of condensate formation [6]. This suggests that, while BEC's formed through stimulated emission may have a low temperature relative to the center of mass, the center-of-mass motion may itself have a higher temperature. Current experimental measurements of single-atom velocity distributions [12] appear insensitive to the center-of-mass temperature of the condensate. The effect is analogous to the increased uncertainty in the frequency and pointing stability of a multimode optical laser, compared to a single-mode laser.

We emphasize that the asymptotic solutions given here are only applicable for condensate nuclei that have already formed, as the spontaneous-emission noise is omitted. The gain medium is assumed to be unsaturated and to equilibrate rapidly, giving a uniform gain constant across the growing condensate. In addition, one can expect damping to occur due to interactions with noncondensed atoms [13], which are not treated here. These interactions, however, can only equilibrate the temperatures of condensed and noncondensed center-of-mass motion, rather than providing additional cooling of the condensate center-of-mass motion. Other damping mechanisms that will intrinsically be present in a BEC are two-body losses, which effectively change g to $g-\Gamma|\Psi|^2$, where Γ is the two-body loss rate. While these affect the form of the asymptotic solution, two-body (or higher-order) losses do not change the amplitude of the condensate center-of-mass oscillation.

In summary, we have found an asymptotic solution to the Gross-Pitaevskii equation with gain, which has the advantage of yielding an explicit analytic result of great physical transparency. The solution shows that the nonequilibrium behavior of a growing Bose-Einstein condensate generally includes an outward momentum component and spatial oscillations. As a result, we suggest that the state of a trapped BEC is not a canonical ensemble, and should be characterized by at least two distinct temperatures: one for the internal degrees of freedom and one for the center-of-mass motion.

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