

# Squeezing in intracavity four-wave mixing with non-degenerate pumps

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**Abstract.** We present a quantum analysis of four-wave mixing in a non-resonant non-linear medium inside a cavity which is driven by two driving fields. In this process a non-linear medium couples two pump modes with a signal mode of the frequency equal to the half-sum of the two pump field frequencies. Three types of stable steady-state solutions of the deterministic problem, describing three regimes of oscillation, are found: one below threshold and two different above-threshold regimes. The spectra of fluctuations of the quadrature-phase amplitudes for all three modes are calculated. It is found that in the above-threshold regime each of three modes may display a squeezed noise reduction. Below threshold a squeezing is realized for the central mode.

## 1. Introduction

One of the important schemes to generate a squeezed light, realized experimentally, is based on the process of non-degenerate four-wave mixing (FWM) in a cavity (see [1] and references therein). In this process an intense pump field with frequency  $\omega$  (for certainty) interacts via a non-linear medium with two weak modes of radiated field with frequencies  $\omega_1$  and  $\omega_2$  such as  $2\omega = \omega_1 + \omega_2$ .

A quantum theory of non-degenerate FWM, based on a resonant interaction of a two-level atomic system with three modes of a radiation field in a cavity, within the undepleted pump approximation in the below-threshold regime has been developed in [2, 3]. The spectrum of two-mode squeezing in the cavity output field for this system has been calculated in [2]. In [4, 5] the squeezing spectrum is calculated for the case of a phenomenological description of a non-linear medium in the non-resonant case. A quantum theory of non-degenerate FWM in an atomic system in the above-threshold regime has been considered in [6] with the use of the method of linearization of the stochastic equations of motion. However, the steady-state solutions for each of the phases of generated modes with frequencies  $\omega_1$  and  $\omega_2$  do not exist in this case. Therefore, the calculation of the spectrum of the two-mode quadrature-amplitude fluctuations with the use of linearization about these phases is impossible above threshold.

In [7, 8] another possibility of squeezed-state generation in intracavity FWM in a resonant atomic system driven by a bichromatic (two-component) pump field is proposed. In this process two pump fields with equal amplitudes and frequencies  $\omega_1$  and  $\omega_2$ , equally detuned from the atomic transition frequency  $\omega_0$ , excite a cavity-

resonant mode with frequency  $\omega_0$  in a squeezed state. This process has a series of peculiarities. As opposed to the standard scheme of non-degenerate FWM with a single pump, one-mode squeezed light is obtained in this case. However, the calculations show [7, 8] that in this process the coefficient of parametric coupling between the conjugate modes is equal to zero and the squeezing is determined only by the spontaneous noise correlators. The vanishing of the coupling coefficient is specific for the configuration of a two-component pump field with equal amplitudes and symmetrical detunings and is related to the mutual cancellation of probabilities of multiphoton processes of radiation and absorption in the two-level atomic medium. There is also an absorption of the  $\omega_0$  mode by the atomic medium. Both these phenomena degrade the squeezing which reaches 35% in this process. In addition, the results of [7, 8] are obtained within the range of a classical pump field and they correspond to the undepleted pump approximation (i.e. to the below-threshold regime).

In the present paper the possibility of one-mode squeezed light generation in the process of FWM in a cavity influenced by two pump fields with equal amplitudes and different frequencies and phases in the below- and above-threshold regimes is studied. As a result of this process two photons of pump fields with frequencies  $\omega_1$  and  $\omega_2$  transform to two photons of a signal field with frequency  $\omega_0 = \frac{1}{2}(\omega_1 + \omega_2)$ . We consider the case when the frequencies of the three cavity modes  $\omega_0$ ,  $\omega_1$  and  $\omega_2$  are non-resonant with respect to any atomic transitions in the non-linear medium. In this case the spontaneous atomic transitions may be ignored and the medium may be described phenomenologically by the third-order susceptibility  $\chi^{(3)}$ . The pump modes  $\omega_1$ ,  $\omega_2$  and the signal mode  $\omega_0$  are described in a quantum way and the pump depletion is taken into account.

It is worth noting that, as opposed to the standard scheme of non-degenerate FWM with a single pump [2, 4, 6], for the present problem three types of stable steady-state solutions may be found for the intensities and phases of all three modes, which correspond to the three regimes of oscillation: one below-threshold and two above-threshold regimes. This fact makes it possible to use the method of linearization for both below- and above-threshold regimes and, in particular, to obtain an analytical expression for the fluctuation spectra of quadrature-phase amplitudes for all three modes in all generation regimes. The calculations show that above threshold, side by side with the squeezing of the  $\omega_0$  mode, there is also a squeezing of quadrature-phase fluctuations for each of the pump modes.

The paper is constructed in the following manner: in section 2 the stochastic equations of motion describing the cavity-modes dynamics are written and their steady-state solutions are obtained. In section 3 the results on squeezing of the  $\omega_0$  mode in the below-threshold regime are given. Section 4 is devoted to the linearization of the equations of motion in the above-threshold regimes and, finally, in section 5 the squeezing spectra for the  $\omega_0$ ,  $\omega_1$  and  $\omega_2$  modes are calculated and analysed.

## 2. Equations of motion and steady states

We consider the following model of parametric four-wave mixing oscillation. A non-linear medium placed inside a suitably tuned ring cavity couples two pump modes of frequencies  $\omega_1$  and  $\omega_2$  with a signal mode of frequency  $\omega_0$ , such that  $\omega_1 + \omega_2 = 2\omega_0$  ( $k_1 + k_2 = 2k_0$ ). The pump modes are driven by two external driving fields. All three

cavity modes are damped via cavity losses. This system may be described by the following Hamiltonian

$$\begin{aligned}
 H &= \sum_{k=1}^4 H_k & H_1 &= \sum_{j=0}^2 \hbar \omega_j a_j^\dagger a_j & H_2 &= \frac{1}{2} i \hbar \kappa (a_1 a_2 a_0^{\dagger 2} - a_1^\dagger a_2^\dagger a_0^2) \\
 H_3 &= i \hbar \sum_{k=1}^2 (E_k e^{-i \omega_k t} a_k^\dagger - E_k^* e^{i \omega_k t} a_k) & H_4 &= \sum_{j=0}^2 (a_j^\dagger \Gamma_j + a_j \Gamma_j^\dagger)
 \end{aligned} \tag{1}$$

where  $a_j^\dagger$ ,  $a_j$  are the boson creation and annihilation operators of the cavity modes  $\omega_j$  ( $j=0, 1, 2$ ), respectively.  $H_1$  is the free part of the Hamiltonian and  $H_2$  describes the effective interaction with the coupling constant  $\frac{1}{2}\kappa$  proportional to the third-order susceptibility. The pumping of modes  $\omega_1$ ,  $\omega_2$  is described by  $H_3$ , where  $E_1$ ,  $E_2$  are proportional to the amplitudes of two coherent phase-locked driving fields of frequencies  $\omega_1$  and  $\omega_2$ . For simplicity we consider all cavity detunings to be zero.  $H_4$  accounts for the decay of the cavity modes, where  $\Gamma_j$ ,  $\Gamma_j^\dagger$  are reservoir operators which will give rise to the cavity damping constants  $\gamma_0$ ,  $\gamma_1$ ,  $\gamma_2$  for the cavity modes  $\omega_0$ ,  $\omega_1$ ,  $\omega_2$ , respectively, and we assume that all the damping constants are small compared with the cavity mode frequencies and the mode spacings.

We follow the standard procedure (see, e.g., [9, 10]) to obtain an interaction picture equation of motion for the reduced density operator of three modes. This equation is

$$\begin{aligned}
 \frac{\partial \rho}{\partial t} &= \frac{1}{2} \kappa [(a_1 a_2 a_0^{\dagger 2} - a_1^\dagger a_2^\dagger a_0^2), \rho] + [(E_1 a_1^\dagger - E_1^* a_1), \rho] + [(E_2 a_2^\dagger - E_2^* a_2), \rho] \\
 &+ \sum_{j=0}^2 \gamma_j (2 a_j \rho a_j^\dagger - \rho a_j^\dagger a_j - a_j^\dagger a_j \rho)
 \end{aligned} \tag{2}$$

where the bath has been assumed to be at zero temperature and we have transformed to the rotating frames

$$a_j(t) \rightarrow a_j e^{-i \omega_j t} \quad a_j^\dagger \rightarrow a_j^\dagger e^{i \omega_j t} \quad (j=0, 1, 2). \tag{3}$$

Then we may transform this operator master equation into a  $c$ -number Fokker-Planck equation using the positive  $p$ -representation [10, 11] with independent complex field variables  $a_j^\dagger$ ,  $a_j$  which correspond to the slowly varying operators  $a_j^\dagger$ ,  $a_j$ . The Fokker-Planck equation derived is equivalent to the following set of stochastic differential equations

$$\begin{aligned}
 \dot{\alpha}_0(t) &= -\gamma_0 \alpha_0 + \kappa \alpha_1 \alpha_2 \alpha_0^\dagger + R_0(t) \\
 \dot{\alpha}_1(t) &= -\gamma_1 \alpha_1 - \frac{1}{2} \kappa \alpha_0^2 \alpha_2^\dagger + E_1 + R_1(t) \\
 \dot{\alpha}_2(t) &= -\gamma_2 \alpha_2 - \frac{1}{2} \kappa \alpha_0^2 \alpha_1^\dagger + E_2 + R_2(t)
 \end{aligned} \tag{4}$$

and the corresponding 'complex conjugate' equations, obtained by exchanging  $\alpha_j \leftrightarrow \alpha_j^\dagger$ ,  $R_j \rightarrow R_j^\dagger$  and terms  $E_{1,2}$  with their complex conjugates.  $R_j^\dagger$ ,  $R_j$  are the Gaussian noise terms with zero means and the following correlations

$$\langle R_0(t) R_0(t') \rangle = \kappa \alpha_1 \alpha_2 \delta(t-t') \quad \langle R_0^\dagger(t) R_0^\dagger(t') \rangle = \kappa \alpha_1^\dagger \alpha_2^\dagger \delta(t-t') \tag{5}$$

$$\langle R_1(t) R_2(t') \rangle = -\frac{1}{2} \kappa \alpha_0^2 \delta(t-t') \quad \langle R_1^\dagger(t) R_2^\dagger(t') \rangle = -\frac{1}{2} \kappa \alpha_0^{\dagger 2} \delta(t-t'). \tag{6}$$

All the other correlators are zero.

Below we will consider the case of equal amplitudes and arbitrary phases of the driving fields

$$|E_1| = |E_2| = E \quad E_k = Ee^{i\phi_k} \quad (k = 1, 2) \tag{7}$$

and the case of equal damping constants  $\gamma_1 = \gamma_2 = \gamma$  for the pump modes  $\omega_1$  and  $\omega_2$ .

The set of equations (4) will be solved by linearization about the steady-state solutions  $\alpha_j^0 = |\alpha_j^0| \exp(i\Psi_j^0)$  ( $(\alpha_j^0)^* = (\alpha_j^0)^+$ ) of the deterministic problem obtained by setting  $\dot{\alpha}_j = R_j = 0$ . For validity of this method it is necessary that the steady state be stable with respect to small fluctuations. The analysis of the steady-state solutions along with the stability conditions (see below) results in three possible regimes of oscillation.

(i) In the region below the generation threshold  $\varepsilon < 1$ ,

$$\varepsilon = E/E_t \quad E_t = \gamma(\gamma_0/\kappa)^{1/2} \tag{8}$$

where  $E_t$  is the threshold value of  $E$ , the stable steady-state solution of set (4) is

$$\alpha_0^0 = 0 \quad |\alpha_1^0| = |\alpha_2^0| = E/\gamma \quad \Psi_1^0 = \phi_1 \quad \Psi_2^0 = \phi_2. \tag{9}$$

In the above-threshold region ( $\varepsilon > 1$ ) the two types of steady-state solutions must be differentiated.

(ii) for the first, the intensities (in photon number units) of the pump modes are equal  $|\alpha_1^0|^2 = |\alpha_2^0|^2$  and the steady-state solutions have the following form

$$|\alpha_0^0| = \left( \frac{2\gamma}{\kappa} (\varepsilon - 1) \right)^{1/2} \tag{10}$$

$$|\alpha_1^0| = |\alpha_2^0| = (\gamma_0/\kappa)^{1/2} \tag{11}$$

$$\Psi_1^0 = \phi_1 \quad \Psi_2^0 = \phi_2 \quad 2\Psi_0^0 = \phi_1 + \phi_2. \tag{12}$$

This solution is stable for the region  $1 < \varepsilon < 2$ .

(iii) For the second solution  $|\alpha_1^0| \neq |\alpha_2^0|$  and we have

$$|\alpha_0^0| = (2\gamma/\kappa)^{1/2} \tag{13}$$

$$|\alpha_1^0| = \frac{1}{2}(\gamma_0/\kappa)^{1/2}[\varepsilon + (\varepsilon^2 - 4)^{1/2}] \quad |\alpha_2^0| = \frac{1}{2}(\gamma_0/\kappa)^{1/2}[\varepsilon - (\varepsilon^2 - 4)^{1/2}] \tag{14a}$$

or

$$|\alpha_1^0| = \frac{1}{2}(\gamma_0/\kappa)^{1/2}[\varepsilon - (\varepsilon^2 - 4)^{1/2}] \quad |\alpha_2^0| = \frac{1}{2}(\gamma_0/\kappa)^{1/2}[\varepsilon + (\varepsilon^2 - 4)^{1/2}] \tag{14b}$$

$$\Psi_1^0 = \phi_1 \quad \Psi_2^0 = \phi_2 \quad 2\Psi_0^0 = \phi_1 + \phi_2 \tag{15}$$

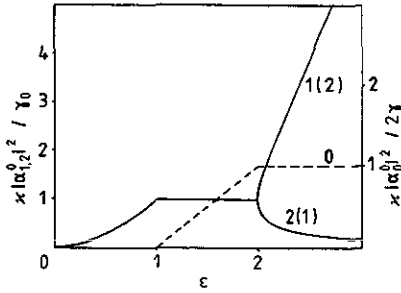
and in this case the set is stable at  $\varepsilon > 2$ .

Note that for all the stationary amplitudes  $|\alpha_1^0|, |\alpha_2^0|$  above threshold the following relation is met

$$|\alpha_1^0| |\alpha_2^0| = E_t^2/\gamma^2 \quad (\varepsilon > 1) \tag{16}$$

i.e. the product of intensities of pump fields inside the cavity saturates. For comparison, below threshold we have  $|\alpha_1^0| |\alpha_2^0| = E^2/\gamma^2$ .

The dependence of the steady-state values of the intensities of three modes inside the cavity on the ratio  $\varepsilon = E/E_t$  is plotted in figure 1. Beginning at the threshold value  $E_t$  the photon number in the mode  $\omega_0$  increases linearly with a corresponding increase of  $\varepsilon$ , and the pump-mode photon number is constant in this case. The values  $\varepsilon = 1, 2$  are instability points, since all the steady states are unstable at these points. It can be



**Figure 1.** The dependence of the scaled steady-state intensities on the parameter  $\epsilon = E/E_1$ . The broken line (and the right-hand scale) represents the signal mode intensity  $\kappa|\alpha_0^0|^2/(2\gamma)$ . The full lines represent the pump mode intensities  $\kappa|\alpha_{1,2}^0|^2/\gamma_0$ . In accordance with the bistable behaviour of the pump intensities in the region  $\epsilon > 2$ , the upper curve may correspond to  $|\alpha_1^0|^2$ , then the lower curve corresponds to  $|\alpha_2^0|^2$ , and vice versa.

easily verified that, at  $\epsilon = 2$ , the mean photon number in units of cavity lifetime in the mode  $\omega_0$  reaches the sum of the corresponding mean photon number of the two pump modes

$$\gamma_0|\alpha_0^0|^2 = \gamma|\alpha_1^0|^2 + \gamma|\alpha_2^0|^2 \quad (\epsilon = 2). \tag{17}$$

In the region  $\epsilon > 2$  the steady-state amplitudes of the modes  $\omega_1, \omega_2$  have a bistable behaviour, according to (14a, b). In this region the creation of photons of the  $\omega_0$  mode is cancelled by the cavity losses as well as by the reverse process of absorption of photon pairs with frequency  $\omega_0$  and radiation of photons with frequencies  $\omega_1$  and  $\omega_2$ , and the intensity of the  $\omega_0$  mode remains unchanged. The pump modes  $\omega_1, \omega_2$  have an asymmetric behaviour, and for each pair of solutions (14a, b) the following condition is met

$$|\alpha_1^0| + |\alpha_2^0| = E/\gamma \quad (\epsilon > 2). \tag{18}$$

In the steady-state solutions derived attention must also be paid to the phase relations. So, (12) and (15) imply that above threshold the steady-state phases of all three modes turn out to be defined and the solutions (ii) and (iii) are stable. This fact distinguishes the process under consideration from the process of non-degenerate FWM with a single pump field, where only the steady-state phase of the pump mode and the sum of phases of generated signal and idler modes are defined above threshold [4, 6]. Each of the signal and idler mode phases are undefined, therefore the corresponding steady states are unstable and the linearization about these phases is inapplicable above threshold.

### 3. Below threshold results

Let us now turn to the linearization of equations of motion (4) in the below-threshold regime (i). Introducing small fluctuations

$$\Delta\alpha_j(t) = \alpha_j(t) - \alpha_j^0 \quad \Delta\alpha_j^+(t) = \alpha_j^+(t) - \alpha_j^{0*} \tag{19}$$

about the steady states (9), we obtain for the pump modes in the linear approximation

$$\Delta\dot{\alpha}_k(t) = -\gamma\Delta\alpha_k(t) \quad \Delta\dot{\alpha}_k^+(t) = -\gamma\Delta\alpha_k^+(t) \quad (k = 1, 2). \tag{20}$$

For the signal  $\omega_0$  mode  $\Delta\alpha_0(t) = \alpha_0(t)$ ,  $\Delta\alpha_0^+(t) = \alpha_0^+(t)$  and we obtain

$$\frac{\partial}{\partial t} \begin{pmatrix} \alpha_0(t) \\ \alpha_0^+(t) \end{pmatrix} = -\mathbf{A} \begin{pmatrix} \alpha_0 \\ \alpha_0^+ \end{pmatrix} + \begin{pmatrix} R_0^0(t) \\ R_0^{0+}(t) \end{pmatrix} \quad (21)$$

where the drift matrix  $\mathbf{A}$  is

$$\mathbf{A} = \begin{pmatrix} \gamma_0 & -c \\ -c^* & \gamma_0 \end{pmatrix} \quad (22)$$

$c = \kappa\alpha_1^0\alpha_2^0$ , and the non-zero noise correlators are

$$\langle R_0^0(t)R_0^0(t') \rangle = c\delta(t-t') \quad \langle R_0^{0+}(t)R_0^{0+}(t') \rangle = c^*\delta(t-t'). \quad (23)$$

The linearized equations (and hence the steady-state solutions) are stable provided the real parts of the eigenvalues of the drift matrix  $\mathbf{A}$  are positive. Finding these eigenvalues may convince one that the steady-state solutions (9) are stable when  $\varepsilon < 1$ .

We see, that below threshold the pump modes fluctuations  $\Delta\alpha_1$ ,  $\Delta\alpha_2$  are not coupled with the  $\omega_0$ -mode fluctuations. The equations of motion for the  $\omega_0$  mode correspond to the undepleted pump approximation and they turn out to coincide with the well known equations of motion, describing the processes of degenerate parametric oscillation and degenerate FWM below threshold [12, 13]. By analogy with these papers the final results will be written below.

For the mean photon number per unit time close to the frequency  $\omega_0$  at the output of the single-ported cavity we obtain (see also (42))

$$n_0^{\text{out}} = 2\gamma_0 \langle \alpha_0^+(t)\alpha_0(t) \rangle = \varepsilon^2/(1-\varepsilon^4). \quad (24)$$

The spectrum of fluctuations of the quadrature-phase amplitude for the cavity output field centered on the frequency  $\omega_0$  is given by

$$S_0^{\text{out}}(\vartheta_0, t) = 1 + 2\gamma_0 \int_{-\infty}^{+\infty} d\tau e^{-i\omega\tau} \langle :X_0(\vartheta_0, t+\tau), X_0(\vartheta_0, t): \rangle \quad (25)$$

where the following notation is used  $\langle :A, B: \rangle = \langle :AB: \rangle - \langle A \rangle \langle B \rangle$ ,  $::$  denotes normal ordering, and

$$X_0(\vartheta_0, t) = a_0(t)e^{-i\vartheta_0} + a_0^+(t)e^{i\vartheta_0} \quad (26)$$

is the quadrature-phase amplitude operator of the mode  $\omega_0$  mode.

The minimal value of quantity  $S_0^{\text{out}}(\vartheta_0, \omega)$ , which is realized at  $2\vartheta_0 = \phi_1 + \phi_2 + \pi$ , is equal to

$$S_{0, \text{min}}^{\text{out}}(\omega) = 1 - \frac{4\varepsilon^2}{(1+\varepsilon^2)^2 + (\omega/\gamma_0)^2}. \quad (27)$$

Equation (27) describes the squeezing spectrum of the  $\omega_0$  mode ( $0 < S_{0, \text{min}}^{\text{out}} < 1$ ) with the maximal effect (100% squeezing) achieved in the limit  $E \rightarrow E_c$ . It must be remembered, however, that the linearization procedure and the validity of results (24), (27) will break down in the critical region near  $E = E_c$ , where fluctuations become large.

#### 4. Analysis and linearization of equations of motion above threshold

To analyse the set of equations (4) above threshold it is convenient to transform it (following [14]) to the new variables

$$n_j = \alpha_j \alpha_j^+ \quad \Psi_j = \frac{1}{2i} \ln \left( \frac{\alpha_j}{\alpha_j^+} \right) \quad (j=0, 1, 2). \quad (28)$$

Using these variables the set of equations of motion takes the following form

$$\begin{aligned} \dot{n}_0(t) &= -2\gamma_0 n_0 + 2\kappa n_0 \sqrt{n_1 n_2} \cos \Psi + F_0(t) \\ \dot{n}_1(t) &= -2\gamma n_1 - \kappa n_0 \sqrt{n_1 n_2} \cos \Psi + 2\sqrt{n_1} E \cos(\phi_1 - \Psi_1) + F_1(t) \end{aligned} \quad (29)$$

$$\dot{n}_2(t) = -2\gamma n_2 - \kappa n_0 \sqrt{n_1 n_2} \cos \Psi + 2\sqrt{n_2} E \cos(\phi_2 - \Psi_2) + F_2(t)$$

$$\dot{\Psi}_0(t) = \kappa \sqrt{n_1 n_2} \sin \Psi + f_0(t)$$

$$\dot{\Psi}_1(t) = \frac{1}{2} \kappa n_0 \sqrt{n_2/n_1} \sin \Psi + (E/\sqrt{n_1}) \sin(\phi_1 - \Psi_1) + f_1(t) \quad (30)$$

$$\dot{\Psi}_2(t) = \frac{1}{2} \kappa n_0 \sqrt{n_1/n_2} \sin \Psi + (E/\sqrt{n_2}) \sin(\phi_2 - \Psi_2) + f_2(t)$$

where  $\Psi = \Psi_1 + \Psi_2 - 2\Psi_0$ , and the noise terms are

$$F_j = \alpha_j^+ R_j + \alpha_j R_j^+ \quad f_j = \frac{R_j}{2i\alpha_j} - \frac{R_j^+}{2i\alpha_j^+}. \quad (31)$$

The steady-state solutions expressed in new variables are

$$(ii) \quad 1 < \varepsilon < 2: n_0^0 = \frac{2\gamma}{\kappa}(\varepsilon - 1) \quad n_1^0 = n_2^0 = \gamma_0/\kappa \quad (32)$$

$$(iii) \quad \varepsilon > 2: n_0^0 = 2\gamma/\kappa \quad (33)$$

$$n_1^0 = \frac{\gamma_0}{2\kappa} [\varepsilon^2 - 2 + \varepsilon(\varepsilon^2 - 4)^{1/2}] \quad n_2^0 = \frac{\gamma_0}{2\kappa} [\varepsilon^2 - 2 - \varepsilon(\varepsilon^2 - 4)^{1/2}] \quad (34a)$$

or

$$n_1^0 = \frac{\gamma_0}{2\kappa} [\varepsilon^2 - 2 - \varepsilon(\varepsilon^2 - 4)^{1/2}] \quad n_2^0 = \frac{\gamma_0}{2\kappa} [\varepsilon^2 - 2 + \varepsilon(\varepsilon^2 - 4)^{1/2}]. \quad (34b)$$

For both cases (ii) and (iii) the steady-state phases are equal to

$$\Psi_1^0 = \phi_1 \quad \Psi_2^0 = \phi_2 \quad 2\Psi_0^0 = \phi_1 + \phi_2. \quad (35)$$

Introducing small fluctuations

$$\Delta n_j(t) = n_j(t) - n_j^0 \quad \Delta \Psi_j(t) = \Psi_j(t) - \Psi_j^0 \quad (36)$$

about the steady states, we obtain the following sets of linearized equations in the matrix form

$$\Delta \dot{n}(t) = -A \Delta n + F(t) \quad (37)$$

$$\Delta \dot{\Psi}(t) = -\tilde{A} \Delta \Psi + f(t) \quad (38)$$

where  $\Delta n = (\Delta n_0, \Delta n_1, \Delta n_2)^T$ ,  $\Delta \Psi = (\Delta \Psi_0, \Delta \Psi_1, \Delta \Psi_2)^T$ ,  $F = (F_0^0, F_1^0, F_2^0)^T$ ,  $f = (f_0^0, f_1^0, f_2^0)^T$  and the drift matrices are

$$\mathbf{A} = \begin{pmatrix} 0 & -\frac{\gamma_0 n_0^0}{n_1^0} & -\frac{\gamma_0 n_0^0}{n_2^0} \\ \gamma_0 & \gamma & \frac{\gamma_0 n_0^0}{2n_2^0} \\ \gamma_0 & \frac{\gamma_0 n_0^0}{2n_1^0} & \gamma \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} 2\gamma_0 & -\gamma_0 & -\gamma_0 \\ \frac{\gamma_0 n_0^0}{n_1^0} & \gamma & -\frac{\gamma_0 n_0^0}{2n_1^0} \\ \frac{\gamma_0 n_0^0}{n_2^0} & -\frac{\gamma_0 n_0^0}{2n_2^0} & \gamma \end{pmatrix}. \quad (39)$$

In these equations the noise terms  $F_j^0(t)$ ,  $f_j^0(t)$  are obtained from (31) by substitution of the corresponding steady-state solutions (ii) or (iii). The non-zero correlations of these noises are

$$\langle F_0^0(t) F_0^0(t') \rangle = 2\gamma_0 n_0^0 \delta(t-t') \quad \langle F_1^0(t) F_2^0(t') \rangle = -\gamma_0 n_0^0 \delta(t-t') \quad (40)$$

$$\langle f_0^0(t) f_0^0(t') \rangle = -\frac{\gamma_0}{2n_0^0} \delta(t-t') \quad \langle f_1^0(t) f_2^0(t') \rangle = \frac{\kappa^2 n_0^0}{4\gamma_0} \delta(t-t'). \quad (41)$$

The linearization procedure is valid if the steady-state solutions are stable, i.e. if the real parts of all the eigenvalues of the matrices  $-\mathbf{A}$ ,  $-\bar{\mathbf{A}}$  are negative. Using the Hurwitz criteria it may be obtained that, as mentioned earlier, the steady-state solutions (ii) and (iii) are stable in the regions  $1 < \varepsilon < 2$  and  $\varepsilon > 2$ , respectively. The values  $\varepsilon = 1, 2$  are instability points.

The mean photon number per unit time for the cavity-output field close to the frequency  $\omega_j$  is (see, e.g., [13])

$$n_j^{\text{out}}(t) = 2\gamma_j \langle \alpha_j^+(t) \alpha_j(t) \rangle + n_j^{\text{in}} \quad (42)$$

where  $n_j^{\text{in}}$  is the corresponding input photon number ( $j=0, 1, 2$ ;  $\gamma_1 = \gamma_2 = \gamma$ ) and the cavity is assumed to be with a single output port. Taking into account that for the spontaneously generated  $\omega_0$  mode  $n_0^{\text{in}} = 0$ , and for the pump modes

$$n_1^{\text{in}} = n_2^{\text{in}} = E^2/2\gamma \quad (43)$$

we obtain, in accordance with the linear approximation, that

$$n_0^{\text{out}} = 2\gamma_0 n_0^0 \quad n_{1,2}^{\text{out}} = 2\gamma n_{1,2}^0 + E^2/2\gamma. \quad (44)$$

The presence of fluctuations comes out in the output spectral intensities of the fields with the carrier frequencies  $\omega_j$  in the form of weak and broad band parts as compared with the high-intensity and narrow coherent components proportional to  $n_j^0 \delta(\omega - \omega_j)$ .

The temporal solutions of equations (37), (38) are convenient for analysis of different two- and equal-time correlation functions of amplitudes (intensities) and phases of the fields and, in particular, for analysis of noise in the quadrature-phase amplitudes of three modes in the cavity. However, to calculate the spectral correlations and, in particular, the fluctuation spectra of the quadrature-phase amplitudes for the cavity-output fields it is more convenient to pass into frequency space.

Introducing the Fourier-transformed frequency components of fluctuations

$$\Delta \Psi(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt e^{-i\omega t} \Delta \Psi(t) \quad \Delta n(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt e^{-i\omega t} \Delta n(t) \quad (45)$$



we obtain instead of (37) and (38) the following sets of algebraic equations for fluctuations  $\Delta n(\omega)$ ,  $\Delta\Psi(\omega)$

$$(A + i\omega I)\Delta n(\omega) = F(\omega) \quad (46)$$

$$(\bar{A} + i\omega I)\Delta\Psi(\omega) = f(\omega) \quad (47)$$

where  $I$  is the identity matrix. The non-zero correlations of the Fourier components of the noise terms are

$$\langle F_0^0(\omega)F_0^0(\omega') \rangle = -2\langle F_1^0(\omega)F_2^0(\omega') \rangle = 2\gamma_0 n_0^0 \delta(\omega + \omega') \quad (48)$$

$$\langle f_0^0(\omega)f_0^0(\omega') \rangle = -\frac{\gamma_0}{2n_0^0} \delta(\omega + \omega') \quad \langle f_1^0(\omega)f_2^0(\omega') \rangle = \frac{\kappa^2 n_0^0}{4\gamma_0} \delta(\omega + \omega'). \quad (49)$$

Solving the sets of equations (46) and (47) and using the correlators (48) and (49) one can obtain all kinds of second-order expectation values for the fluctuations  $\Delta n_j(\omega)$ ,  $\Delta\Psi_j(\omega)$ . In the present paper we restrict ourselves to consideration of such solutions of these sets, which determine the squeezing spectra of quadrature-phase amplitude fluctuations for each of three cavity-output fields.

## 5. Squeezing spectra above threshold

The spectrum of quadrature-phase amplitude fluctuations for the cavity-output field around the frequency  $\omega_j$  is defined by

$$S_j^{\text{out}}(\vartheta_j, \omega) = 1 + 2\gamma_j \int_{-\infty}^{+\infty} d\tau e^{-i\omega\tau} \langle :X_j(\vartheta_j, t + \tau), X_j(\vartheta_j, t): \rangle \quad (50)$$

where

$$X_j(\vartheta_j, t) = a_j(t)e^{-i\vartheta_j} + a_j^\dagger(t)e^{i\vartheta_j} \quad (j=0, 1, 2) \quad (51)$$

is the quadrature-phase amplitude operator of the  $\omega_j$  mode in the cavity.

Using the correspondence between the time-ordered, normally ordered operator averages and  $c$ -number averages [13], it can be easily verified that the minimal value of  $S_j^{\text{out}}(\vartheta_j, \omega)$  is realized at  $\vartheta_j = \Psi_j^0 + \frac{1}{2}\pi$  and in lowest order in small fluctuations is equal to

$$S_{j, \text{min}}^{\text{out}}(\omega) = 1 + 8\gamma_j n_j^0 \int_{-\infty}^{+\infty} d\tau e^{-i\omega\tau} \langle \Delta\Psi_j(t + \tau)\Delta\Psi_j(t) \rangle. \quad (52)$$

This expression may be written in terms of Fourier components as follows

$$S_{j, \text{min}}^{\text{out}}(\omega) = 1 + 8\gamma_j n_j^0 \int_{-\infty}^{+\infty} d\omega' e^{i(\omega + \omega')\tau} \langle \Delta\Psi_j(\omega)\Delta\Psi_j(\omega') \rangle. \quad (53)$$

The term unity in (53) gives the vacuum level of fluctuations, and the squeezing is realizable if  $S_{j, \text{min}}^{\text{out}}(\omega) < 1$ .  $S_{j, \text{min}}^{\text{out}}(\omega) = 0$  implies a perfect (100%) squeezing.

Thus, to calculate the output squeezing spectra (53) for the  $\omega_j$  modes it is sufficient to calculate the averages  $\langle \Delta\Psi_j(\omega)\Delta\Psi_j(\omega') \rangle$ , i.e. we can restrict ourselves with the solutions of the set of equations (47). These solutions are

$$\begin{aligned} \Delta\Psi_0(\omega) = & \frac{(\gamma + i\omega)^2 - \gamma_0^2 n_0^{02}/4n_1^0 n_2^0}{D(\omega)} f_0^0(\omega) + \frac{\gamma_0^2 n_0^0/(2n_2^0) + \gamma_0(\gamma + i\omega)}{D(\omega)} f_1^0(\omega) \\ & + \frac{\gamma_0^2 n_0^0/(2n_1^0) + \gamma_0(\gamma + i\omega)}{D(\omega)} f_2^0(\omega) \end{aligned} \quad (54)$$

$$\begin{aligned} \Delta\Psi_{1(2)}(\omega) = & -\frac{2\gamma_0^2 n_0^{02}/(4n_1^0 n_2^0) + 2\gamma_0(\gamma + i\omega)n_0^0/2n_1^0}{D(\omega)} f_0^0(\omega) \\ & + \frac{(2\gamma_0 + i\omega)(\gamma + i\omega) + 2\gamma_0^2 n_0^0/2n_{2(1)}^0}{D(\omega)} f_{1(2)}^0(\omega) \\ & + \frac{\gamma_0(2\gamma_0 + i\omega)n_0^0/2n_{1(2)}^0 - 2\gamma_0^2 n_0^0/2n_{1(2)}^0}{D(\omega)} f_{2(1)}^0(\omega) \end{aligned} \quad (55)$$

where

$$\begin{aligned} D(\omega) = \det \bar{A} = & (2\gamma_0 + i\omega)(\gamma + i\omega)^2 - \gamma_0^2 n_0^{02}(2\gamma_0 + i\omega)/(4n_1^0 n_2^0) \\ & + 2\gamma_0^2(\gamma + i\omega)(n_0^0/2n_1^0 + n_0^0/2n_2^0) + 4\gamma_0^3 n_0^{02}/4n_1^0 n_2^0. \end{aligned} \quad (56)$$

On the basis of results (54) and (55) and with use of correlators (49) we obtain the following second-order expectation values

$$\begin{aligned} \langle \Delta\Psi_0(\omega)\Delta\Psi_0(\omega') \rangle = & \left( 2q_0^2 \frac{1 + q_1 q_2 + (q_1 + q_2) + r^2(\omega/\gamma_0)^2}{\gamma_0 n_0^0 d(\omega)} \right. \\ & \left. - \frac{[q_1 q_2 - 1 + r^2(\omega/\gamma_0)^2] + 4r^2(\omega/\gamma_0)^2}{2\gamma_0 n_0^0 d(\omega)} \right) \delta(\omega + \omega') \end{aligned} \quad (57)$$

$$\begin{aligned} \langle \Delta\Psi_k(\omega)\Delta\Psi_k(\omega') \rangle = & \left( \frac{q_0^2(1 + 2r)(\omega/\gamma_0)^2}{\gamma n_k^0 d(\omega)} \right. \\ & \left. - \frac{(q_1 q_2 + q_k)^2 + r^2 q_k^2 (\omega/\gamma_0)^2}{\gamma n_k^0 q_k d(\omega)} \right) \delta(\omega + \omega') \quad (k = 1, 2) \end{aligned} \quad (58)$$

where the following notations are introduced for convenience

$$r = \gamma_0/\gamma \quad q_0 = \kappa n_0^0/2\gamma \quad q_{1,2} = n_0^0 \gamma_0/2n_{1,2}^0 \gamma \quad (59)$$

and

$$\begin{aligned} d(\omega) = & \frac{|D(\omega)|^2}{\gamma^6 r^2} = 4[1 + q_1 q_2 + q_1 + q_2 - (r + r^2)(\omega/\gamma_0)^2]^2 \\ & + (\omega/\gamma_0)^2 [q_1 q_2 - 2r(q_1 + q_2) - 4r - 1 + r^2(\omega/\gamma_0)^2]^2. \end{aligned} \quad (60)$$

Note, the quantity  $d(\omega)$  is always non-zero.

Substituting the expressions (57), (58) into (53) we obtain the following results for

the squeezing spectra of the  $\omega_0$  and  $\omega_k$  ( $k = 1, 2$ ) modes, respectively

$$S_{0,\min}^{\text{out}}(\omega) = 1 + 16q_0^2 \frac{1 + q_1q_2 + q_1 + q_2 + r^2(\omega/\gamma_0)^2}{d(\omega)} - 4 \frac{[q_1q_2 - 1 + r^2(\omega/\gamma_0)^2]^2 + 4r^2(\omega/\gamma_0)^2}{d(\omega)} \quad (61)$$

$$S_{k,\min}^{\text{out}}(\omega) = 1 + \frac{8q_0^2(1 + 2r)(\omega/\gamma)^2}{r^2d(\omega)} - \frac{8(q_1q_2 + q_k)^2 + 8q_k^2(\omega/\gamma)^2}{q_kd(\omega)}. \quad (62)$$

The results (61) and (62) are represented in general form and they describe the squeezing spectra in both above-threshold regimes (ii) and (iii). For each of these regimes the values of  $n_j^0$  are different and the quantities  $q_k$  are respectively equal to

$$(ii) \quad 1 < \varepsilon < 2: \quad q_0 = q_1 = q_2 = \varepsilon - 1 \quad (63)$$

$$(iii) \quad \varepsilon > 2: \quad q_0 = 1 \quad q_{1,2} = \frac{2}{\varepsilon^2 - 2 \pm \varepsilon\sqrt{\varepsilon^2 - 4}} \quad (64a)$$

for solutions (34a), or

$$q_0 = 1 \quad q_{1,2} = \frac{2}{\varepsilon^2 - 2 \mp \varepsilon\sqrt{\varepsilon^2 - 4}} \quad (64b)$$

for solutions (34b). Thus, the choice of the solutions (34a) or (34b) results merely in the interchange of the index ( $k = 1$  or  $2$ ) in expression (62).

The results (61) and (62) are obtained within the ranges of the linearization method and are valid if the conditions of fluctuation smallness are met

$$\langle(\Delta n_j)^2\rangle^{1/2} \ll n_j^0 \quad | \langle(\Delta \Psi_j)^2\rangle | \ll 1. \quad (65)$$

It is evident from physical considerations that these conditions are obviously met for intensive fields  $n_j^0 \gg 1$ . In the region  $1 < \varepsilon < 2$  these conditions are reduced to:  $(\gamma/\kappa)(\varepsilon - 1) \gg 1$ ,  $\gamma_0/\kappa \gg 1$ ; and in the well above-threshold region ( $\varepsilon \gg 2$ ) they are  $\gamma/\kappa \gg 1$ ,  $\gamma_0/\kappa\varepsilon^2 \gg 1$  (see also figure 1).

The analysis of expressions (61) and (62), represented in graphical form in figures 2–6, shows that in the above-threshold regime there is significant squeezing both for the signal  $\omega_0$  mode (100% squeezing as a possible limit) and for each of the pump modes  $\omega_1$  and  $\omega_2$  (50% squeezing). This fact is an interesting peculiarity of the considered process of FWM with two pump fields. It will be recalled that in the usual non-degenerate FWM with a single pump only two-mode squeezed states for the combined field at corresponding frequencies of the side-band modes are formed.

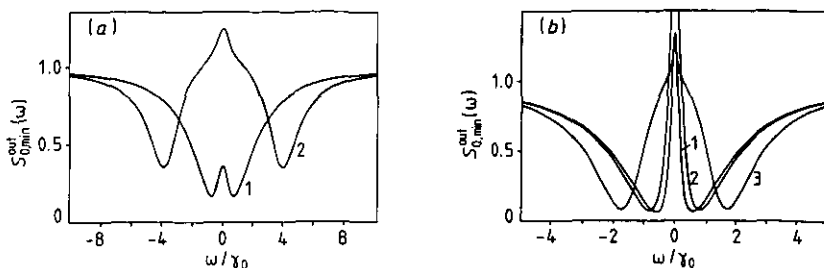
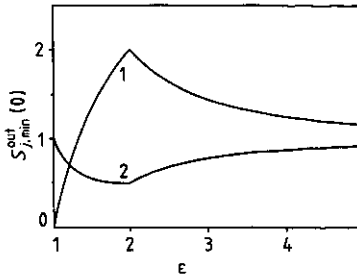


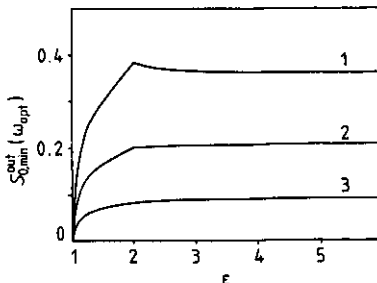
Figure 2. The squeezing spectrum  $S_{0,\min}^{\text{out}}(\omega)$  of the signal field plotted against  $\omega/\gamma_0$  for different values of parameters  $r$  and  $\varepsilon$ : (a)  $r = 2$ ,  $\varepsilon = 1.1$  (curve 1),  $\varepsilon = 4$  (curve 2); (b)  $r = 10$ ,  $\varepsilon = 1.5$  (curve 1),  $\varepsilon = 2.1$  (curve 2),  $\varepsilon = 4$  (curve 3).



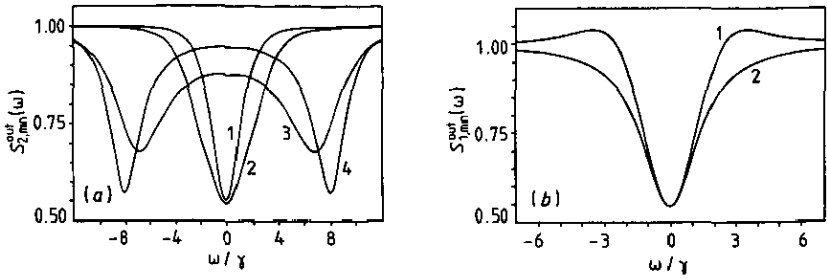
**Figure 3.** The values of the squeezing spectra  $S_{j,\min}^{\text{out}}$  at  $\omega = 0$  for the signal (curve 1) and pump (curve 2) fields as a function of  $\epsilon$ . These values do not depend on  $r$  and for the two pump fields they are same in the whole region of  $\epsilon$ .

The squeezing spectrum (61) for the signal field around the frequency  $\omega_0$  is plotted in figure 2 for different values of parameters  $\epsilon$  and  $r$ . For  $\epsilon \approx 1$  and small  $r$  the spectrum is single peaked. The value at  $\omega = 0$  depends on  $\epsilon$  only, and a large squeezing in the region of zero frequency occurs near threshold (see curve 1 in figure 3). With increasing  $\epsilon$  the spectrum becomes double peaked, and large squeezing is reached at the side-band frequencies and for large values of the parameter  $r = \gamma/\gamma_0$ . The value of  $S_{0,\min}^{\text{out}}(\omega)$  at the point of minimum  $\omega_{\text{opt}}$  as a function of  $\epsilon$  is plotted in figure 4. It may be seen from figures 2 and 4 that a level of fluctuation suppression close to perfect (100%) squeezing may be reached for values of  $r \geq 10$ .

The squeezing spectra for the pump fields described by (62) are represented in figure 5. Within the range  $1 < \epsilon < 2$ , where the intensities of the two pump fields are equal to each other, the squeezing spectra for both pump fields coincide and have a single peaked form. The dependence of  $S_{k,\min}^{\text{out}}(0)$  on  $\epsilon$  is represented in figure 3 (curve 2). A large value of squeezing (50% squeezing in the limiting case) is reached near  $\epsilon = 2$  and in the region of zero frequency. In the region  $\epsilon > 2$  the intensities of two pump fields are not equal and the corresponding squeezing spectra, in accordance with (64), become different. A significant squeezing over the whole range  $\epsilon > 2$  occurs only for that pump mode, the intensity of which decreases with the corresponding increase of  $\epsilon$  (see curves 2–4 in figure 5(a)). The spectrum value at  $\omega = 0$  (figure 2, curve 2) increases with increasing  $\epsilon$  and tends to the level of vacuum fluctuations at  $\epsilon \gg 1$ . However, for this pump field there is always a region of moderate values of  $r$  (see figure 6 which represents the value of the squeezing spectrum for the low-intensity pump field at the point of minimum  $\omega_{\text{opt}}$  as a function of parameter  $r$ ), when the squeezing spectrum has a double-peaked structure and squeezing is maximal at the



**Figure 4.** The value of  $S_{0,\min}^{\text{out}}$  at the frequency  $\omega_{\text{opt}}$ , when the squeezing of the signal field is maximal, as a function of  $\epsilon$  for  $r = 2$  (curve 1),  $r = 4$  (curve 2) and  $r = 10$  (curve 3).

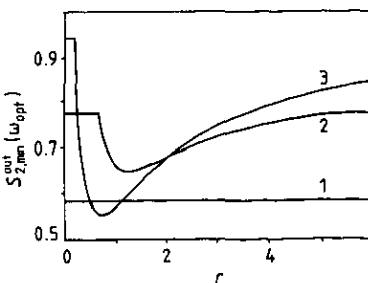


**Figure 5.** The squeezing spectra  $S_{k,\min}^{\text{out}}(\omega)$  of the pump fields plotted against  $\omega/\gamma$ . (a) For both pump fields: curve 2,  $\epsilon = 2.1$ ,  $r = 2$ ; curve 3,  $\epsilon = 4$ ,  $r = 2$ ; curve 4,  $\epsilon = 6$ ,  $r = 1$ , correspond to the low-intensity pump field ( $k = 2$ ), and curve 1,  $\epsilon = 1.5$ ,  $r = 2$ , is related to the regime (ii) with equal pump intensities. (b) Corresponds to the high-intensity pump field ( $k = 1$  for certainty):  $\epsilon = 2.1$ ,  $r = 2$  (curve 1),  $r = 10$  (curve 2).

side-band frequencies. The largest effect tends to 50% for such values of  $r$  and large  $\epsilon$ . As far as the pump mode of increasing intensity is concerned in the region  $\epsilon > 2$ , the corresponding squeezing spectrum (figure 5(b)) is always single peaked. Accordingly the maximal squeezing and an optimal region of values of  $\epsilon$  for this pump field are determined by the spectrum value at the  $\omega = 0$  point (figure 2, curve 2).

**6. Summary**

In conclusion we enumerate the main characteristics of intracavity FWM under the influence of two external driving fields. An essential feature of this problem is that it permits an analytical consideration within the framework of a well developed method of linearization of stochastic equations of motion with allowance for pump depletion. As a result a quantum analysis of amplitude and phase fluctuations in both regimes of oscillation, below and above threshold, is carried out. It is shown that in such a process the one-mode squeezed states are obtainable in all generation regimes. Below threshold there is only the signal  $\omega_0$  mode which is squeezed. The signal mode remains in squeezed state above threshold in the whole region of intensities of the incident driving fields  $n^{\text{in}} \equiv E^2/(2\gamma) > \gamma\gamma_0/(2\kappa)$  ( $\epsilon > 1$ ) with the maximal efficiency (100% squeezing) realizable for large values of the ratio  $\gamma_0/\gamma$ . Besides, in the above-threshold regime there is a squeezing of both pump fields, which may reach up to 50%. An optimal region for squeezing of the pump fields is  $\gamma\gamma_0/(2\kappa) < n^{\text{in}} < 2\gamma_0\gamma/\kappa$  ( $1 < \epsilon < 2$ ),



**Figure 6.** The dependence of the spectrum value  $S_{2,\min}^{\text{out}}(\omega_{\text{opt}})$  for the low-intensity pump field on parameter  $r$ :  $\epsilon = 2.2$  (curve 1),  $\epsilon = 3$  (curve 2),  $\epsilon = 6$  (curve 3).

where their intensities are equal. In the region  $n^m > 2\gamma_0\gamma/\kappa$  ( $\varepsilon > 2$ ) a large squeezing is realized for that pump field, whose intensity decreases with increasing  $n^m$ . The other peculiarity of this process, which deserves special consideration, is the bistable behaviour of the steady-state intensities of the pump modes beginning at the intensity of the driving fields equal to  $n^m = 2\gamma_0\gamma/\kappa$  ( $\varepsilon = 2$ ).

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