SUPERSYMMETRY AND EXACT SOLUTIONS FOR QUANTUM PHASE TRANSITIONS IN RANDOM SPIN CHAINS

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OUTLINE

- Quantum phase transitions in disordered systems exhibit rich new physics.
- A tractable model to study is the random transverse field Ising chain.
- An exact analytic treatment of critical behavior in can be given by mapping it onto a fermion model.
- Supersymmetry and group representation theory plays a key role in solving the associated random Dirac equation. The transfer matrix Hamiltonian can be written in terms of generators of $U(2|1,1)$ or $U(1,1)$.
- The results obtained agree with those recently obtained using a real space renormalisation group technique.

PHASE TRANSITIONS AND CRITICAL PHENOMENA IN DISORDER-FREE SYSTEMS

- Phase transitions in classical statistical mechanics in two dimensions ($d = 2$) are equivalent to quantum phase transitions in $d = 1$.
- Exact solutions based on transfer matrix methods and the Bethe ansatz allow calculation of critical exponents for integrable models.
- Conformal field theory allows a complete classification of all $d = 1$ quantum universality classes.
- Scaling, the renormalisation group, and the $\epsilon = 4 - d$ expansion elucidated universality and non-integrable models.

All of the above methods make essential use of translational invariance.

What about models with random parameters?

A SIMPLE MODEL
TRANSVERSE ISING SPIN CHAIN

$$H_0 = -\sum_{i=1}^{L} \left(J\sigma_i^x \sigma_{i+1}^x + h\sigma_i^z \right)$$

where σ_i^x and σ_i^z are Pauli spin operators.
Equivalent to the transfer matrix of the two-dimensional Ising model.
At $T = 0$, there is a transition from a ferromagnet to a paramagnet at $h = J$.
This phase transition is driven by quantum fluctuations.
Pfeuty (1970) found an exact solution, calculating all the correlation functions.
However, in an external longitudinal field

$$H = H_0 + B \sum_{i=1}^{L} \sigma_i^x$$

the model *cannot* be solved.

THE RANDOM TRANSVERSE FIELD ISING CHAIN

$$H = -\sum_{i=1}^{L} \left(J_i \sigma_i^x \sigma_{i+1}^x + h_i \sigma_i^z \right)$$

• J_i and h_i are independent random variables.

• Equivalent to the transfer matrix of the two-dimensional McCoy-Wu model.

• Transition from a ferromagnetic to a paramagnetic phase at $\langle \ln h_i \rangle = \langle \ln J_i \rangle$.

• D.S. Fisher (1995) used a real space renormalisation group decimation technique, which he claims is *exact* near the critical point. He calculated the phase diagram, critical exponents, and scaling functions for the magnetisation and correlation functions, *including* in an external longitudinal field.

• This extraordinary achievement is possible because the low-energy physics is controlled by a strong disorder fixed point.

• Fisher's technique has now been applied to random dimerised and anisotropic $S = \frac{1}{2}$ Heisenberg AFM, $S = 1$ Heisenberg AFM, quantum Potts chains, and chains with random spin size. The Potts models have the same critical exponents as the transverse Ising model.

exponent	definition	pure	disorder
α	$\epsilon \sim \Delta^{2-\alpha}$	0^+	n.a.
β	$\langle \sigma_i^x \rangle \sim \Delta^\beta$	$1/8$	$2 - \phi$
ν	$\xi_{av} \sim \Delta^{-\nu}$	1	2
η	$\langle \sigma_r^x \sigma_0^x \rangle \sim r^{1-\eta}$ $(\Delta = 0)$	$5/4$	$\phi - 1$

$\phi = (1 + \sqrt{5})/2$ is the golden mean.

• Fisher's results have been confirmed by numerical calculations (Young and Rieger).

HIGHER DIMENSIONS

Pich *et al.* [PRL **81**, 5916 (1998)] studied the $d = 2$ model using Monte Carlo. Motrunich *et al.* [PRB **61**, 1160 (2000)] used real-space renormalisation to map the $d = 2, 3$ model with strong disorder onto a novel percolation/aggregation process.

Both found that similar physics is exhibited to $d = 1$.
z is non-universal and diverges at the critical point.
There are large differences between average and typical correlation functions.

AN EXACT APPROACH FOR $d = 1$

$$H = -\sum_{i=1}^{L} \left(J_i \sigma_i^x \sigma_{i+1}^x + h_i \sigma_i^z \right)$$

A Jordan-Wigner transformation maps spins onto spinless fermions

$$c_n = \sigma_n^- \exp[i\pi \sum_{m=1}^{n-1} \sigma_m^+ \sigma_m^-]$$

where $\sigma_m^{\pm} \equiv \frac{1}{2}(\sigma_m^x \pm i\sigma_m^y)$

$$H = -\sum_{i=1}^{L} \Big(J_i (c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i)$$
$$+ J_i (c_i^\dagger c_{i+1}^\dagger - c_i c_{i+1}) + h_i (2c_i^\dagger c_i - 1) \Big)$$

This is quadratic in the fermion operators and can be diagonalised by a Bogoliubov transformation. The eigenstates will be non-interacting fermions.

The continuum limit (relative to the Fermi wave vector k_F) is Dirac-type equation

$$H = \int dx \Psi^\dagger \left[i v_F \sigma_3 \frac{\partial}{\partial x} + V(x)\sigma_1 \right] \Psi$$

$\Psi(x = n)^\dagger = (c_n^\dagger, c_n)$ and $v_F = J$ is the Fermi velocity.
$V(x)$ is a random potential with

$$\langle V(x) \rangle = \Delta \qquad \langle V(x)V(x') \rangle = \gamma \delta(x - x')$$

$\Delta = \langle h_i \rangle - J$, measures deviation from criticality. For no disorder $|\Delta|$ is the energy gap.
$\gamma = \langle (h_i - \langle h_i \rangle)^2 \rangle$
The energy $D \equiv \gamma / v_F$ is a measure of the disorder.
$\delta \equiv \frac{|\Delta|}{D}$ is a dimensionless measure of the deviation from criticality.

The continuum limit is valid provided $J \gg |\Delta|, \sqrt{\gamma}$.

EXACT DENSITY OF STATES FROM SUPERSYMMETRY METHOD

Example: Schrödinger equation (Bohr and Efetov, 1982)

$$H\psi = \left[-\frac{\partial^2}{\partial x^2} + V(x)\right]\psi(x) = E\psi(x)$$

where $V(x)$ is a random (white-noise) potential with

$$\langle V(x)\rangle = 0 \qquad \langle V(x)V(y)\rangle = D\delta(x-y).$$

The disorder averaged density of states $\rho(E)$ is related to the one-particle Greens function

$$\rho(E) = \mathrm{Im}\langle G(E,x)\rangle = \mathrm{Im}\langle (x|\frac{1}{H-E}|x)\rangle$$

Set $a = H - E + i\epsilon$. If χ is a fermion (anti-commuting) field

$$\int d\chi d\chi^* \, \chi_i \chi_k \exp(-\chi^* a\chi) = det(a)(a^{-1})_{ik}$$

$$\int d\chi d\chi^* \ \exp(-\chi^* a \chi) = det(a)$$

If S is a boson field

$$\int dS dS^* \exp(-S^* a S) = det(a)^{-1}$$

Let $\Phi^\dagger \equiv (\chi, S)$ be a supersymmetric field. Then

$$\int [D\Phi] \exp(-\Phi^* a \Phi) = 1$$

$$G(E,x) = i \int [D\Phi] \chi(x) \chi(x) \exp\left(-i \int dx L\right)$$

with $[D\Phi] = dS dS^* d\chi d\chi^*$ and the Lagrangian

$$L = \Phi^\dagger [E - i\epsilon - V(x)] \Phi - \frac{d\Phi^\dagger}{dx} \frac{d\Phi}{dx}$$

The disorder average is

$$\langle G(E,x) \rangle = \int dV(x) G(E,x) \exp\left(\frac{-1}{D} \int dx V(x)^2\right)$$

The Lagrangian then becomes

$$L = \Phi^\dagger [E - i\epsilon]\Phi - \frac{d\Phi^\dagger}{dx}\frac{d\Phi}{dx} - iD\left(\Phi^\dagger\Phi\right)^2$$

This one-dimensional supersymmetric field theory can be solved exactly by a transfer-matrix method which maps it onto a Schrödinger-type equation (in superspace) which can be solved analytically. (cf. Feymann's path integral $\equiv$ quantum mechanics).

SUPERSYMMETRY AND THE RANDOM DIRAC EQUATION

Balents and Fisher, PRB **56**, 12970 (1997).
Bocquet, Nucl. Phys. B **546**, 621 (1999).

The transfer Hamiltonian is written in terms of generators of the superalgebra $u(2|1,1)$

$$H = EZ_z + \Delta Z_x + \gamma Z_x^2$$

and acts on an infinite dimensional representation of $U(1,1)$.
There are left and right ground states.

$$H|0>_R = 0 \qquad <0|_L H = 0$$

The disorder-averaged Greens function is

$$< G(E) > = < 0|_L \; S_z \; |0>_R$$

where S_z is the fermionic part of Z_z.

DYNAMICAL CRITICAL EXPONENT z

$$\text{Time} \sim \frac{1}{\text{Energy}} \sim (\text{Length})^z$$

For the pure transverse Ising chain, $z = 1$.
For low energies, the density of states

$$\langle \rho(E) \rangle \sim \frac{1}{\text{Length}} \frac{1}{\text{Energy}} \sim E^{1/z-1}$$

Therefore,

$$z = \frac{1}{2\delta}$$

Note:
(i) z is not universal
(ii) z diverges at the critical point.
The results obtained for z and the low-temperature thermodynamic properties agree with results from the real space renormalisation decimation technique, supporting Fisher's claim that it is exact.

THE CHALLENGES

- Is a complete classification of random quantum universality classes possible? Group theory will help.
- Evaluate the probability distribution for the density of states near a quantum critical point. (We have done this for a quantum wire; cond-mat/0006291).
- Exact evaluation of the magnetisation and correlation functions. Are the exponents related to the golden mean. This will be difficult because these involve non-local fermion operators.