## 2.11 Central Forces

A force $\mathbf{F}(x, y, z)$ is called **central** if it has the form:

$$\mathbf{F} = F(r)\hat{\mathbf{r}} = \frac{F(r)\mathbf{r}}{r}$$

where $\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ and $r = \sqrt{x^2 + y^2 + z^2}$.

Central forces act either towards or away from the origin (depending on their sign) with a magnitude dependent only on the distance the object is away from the origin.

**Note:**

$$\begin{aligned} \nabla r &= \frac{\partial r}{\partial x}\hat{\mathbf{i}} + \frac{\partial r}{\partial y}\hat{\mathbf{j}} + \frac{\partial r}{\partial z}\hat{\mathbf{k}} \\ &= \frac{x}{r}\hat{\mathbf{i}} + \frac{y}{r}\hat{\mathbf{j}} + \frac{z}{r}\hat{\mathbf{k}} \\ &= \frac{1}{r}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \\ &= \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}. \end{aligned}$$

**Lemma**: Let $F(r) = -V'(r)$ for some function $V$. Then the central force $\mathbf{F}(r) = F(r)\hat{\mathbf{r}}$ is conservative with potential function $V(r)$.
**Proof**:

$$\begin{aligned} \mathbf{F}(r) &= -V'(r)\,\hat{\mathbf{r}} = -V'(r)\nabla r \\ &= -V'(r)\frac{\partial r}{\partial x}\,\hat{\mathbf{i}} - V'(r)\frac{\partial r}{\partial y}\,\hat{\mathbf{j}} - V'(r)\frac{\partial r}{\partial z}\,\hat{\mathbf{k}} \\ &= -\frac{\partial V(r)}{\partial x}\hat{\mathbf{i}} - \frac{\partial V(r)}{\partial y}\hat{\mathbf{j}} - \frac{\partial V(r)}{\partial z}\hat{\mathbf{k}} \\ &= -\nabla V(r) \quad \text{(using the chain rule)}. \end{aligned}$$

Therefore all central forces are conservative. (We assume here that the function $\mathbf{F}(r)$ is always integrable.) Hence a particle moving in a central force $\mathbf{F}(r) = -V'(r)\hat{\mathbf{r}}$ has energy

$$E = \frac{1}{2}mv^2 + V(r)$$

which remains constant in time.

## 2.12 Conservation of angular momentum

If we have a particle with mass $m$, velocity $\mathbf{v}$ and position $\mathbf{r}$, then that particle has an **angular momentum** given by:

$$\mathbf{L} = m(\mathbf{r} \times \mathbf{v}).$$

This can also be written as linear momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

where $\mathbf{p}$ is the linear momentum of the particle. If a force $\mathbf{F}$ is exerted on the particle, we have from Newton's second law

$$\begin{aligned} \dot{\mathbf{L}} &= m(\dot{\mathbf{r}} \times \mathbf{v}) + m(\mathbf{r} \times \dot{\mathbf{v}}) \\ &= m(\mathbf{v} \times \mathbf{v}) + m(\mathbf{r} \times \dot{\mathbf{v}}) \\ &= \mathbf{r} \times m\dot{\mathbf{v}} \quad (\text{As } \mathbf{v} \times \mathbf{v} = 0) \\ &= \mathbf{r} \times \mathbf{F}, \end{aligned}$$

which is called the **torque**.

For a *central force*

$$\mathbf{F} = -V'(r)\hat{\mathbf{r}} = \frac{-V'(r)}{r}\mathbf{r},$$

so

$$\begin{aligned} \dot{\mathbf{L}} &= \mathbf{r} \times \mathbf{F} \\ &= \frac{-V'(r)}{r}(\mathbf{r} \times \mathbf{r}) \\ &= \mathbf{0}. \end{aligned}$$

Hence $\mathbf{L}$ is constant in time, so we have **conservation of angular momentum.**

Notes:

$$\begin{aligned} \mathbf{r} \cdot \mathbf{L} &= m\mathbf{r} \cdot (\mathbf{r} \times \mathbf{v}) = 0 \\ \mathbf{v} \cdot \mathbf{L} &= m\mathbf{v} \cdot (\mathbf{r} \times \mathbf{v}) = 0 \end{aligned}$$

That is, $\mathbf{r}$ and $\mathbf{v}$ are both orthogonal to $\mathbf{L}$. Thus a central force restricts movement to a plane, which is perpendicular to the angular momentum $\mathbf{L}$.

## 2.13 Kepler's second law

The angular momentum of a particle is related to the rate at which its position vector sweeps out an area: For $dt$ small,

$$\begin{aligned} dA &\approx \frac{1}{2}|\mathbf{r}(t) \times \mathbf{r}(t+dt)| \\ &= \frac{1}{2}|\mathbf{r}(t) \times (\mathbf{r}(t+dt) - \mathbf{r}(t))| \\ &= \frac{1}{2}|\mathbf{r}(t) \times \frac{(\mathbf{r}(t+dt) - \mathbf{r}(t))}{dt}|\ dt \\ &\approx \frac{1}{2}|\mathbf{r}(t) \times \dot{\mathbf{r}}(t)|\ dt, \end{aligned}$$

which becomes exact as $dt \to 0$. Therefore,

$$\frac{dA}{dt} = \frac{1}{2}|\mathbf{r}(t) \times \dot{\mathbf{r}}(t)| = \frac{|\mathbf{L}|}{2m}.$$

So for a central field of force, where $\mathbf{L}$ is constant in time, the rate at which the position vector sweeps out an area is also constant in time. This result is known as **Kepler's second law** - empirically discovered for planetary motion by Kepler.

**Note**: This law, together with the planar motion of the planets, led Newton to deduce that gravity was a central force. Explicitly for a planet of mass $m$ orbiting about the sun (assumed at origin), the force on the planet is given by

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}}$$

where $M$ is the mass of the sun and $r$ is the distance of the planet from the origin. We shall see later that such a law of force predicts elliptic orbits for the planets. Thus gravity is a central force with potential

$$V(r) = -\frac{K}{r}, \quad K = GMm.$$

## 2.14 Central forces and polar coordinates

We assume that the angular momentum vector $\mathbf{L}$ points in the $z$-direction so the particle moves in the $x-y$ plane. We introduce polar coordinates $r, \theta$ such that

$$x = rcos\theta, \quad y = r\sin\theta$$

and $r = \sqrt{x^2+y^2}$ is the distance of the particle from the origin. Hence

$$\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} = r\hat{\mathbf{r}}$$

where $\hat{\mathbf{r}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}$ is the unit vector in the direction of the particle. Using

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{d\hat{\mathbf{r}}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\hat{\boldsymbol{\theta}},$$

where $\hat{\boldsymbol{\theta}} = \frac{d\hat{\mathbf{r}}}{d\theta} = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}$ is the unit vector int the direction of increasing $\theta$, gives us

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}.$$

**Notes:**
1) $\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{r}} \cdot \hat{\mathbf{k}} = 0$.
2) $\frac{d\hat{\mathbf{r}}}{d\theta} = \hat{\boldsymbol{\theta}} \quad \Rightarrow \quad \frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}\hat{\boldsymbol{\theta}}$.
3) $\frac{d\hat{\boldsymbol{\theta}}}{d\theta} = -\hat{\mathbf{r}} \quad \Rightarrow \quad \frac{d\hat{\boldsymbol{\theta}}}{dt} = \frac{d\hat{\boldsymbol{\theta}}}{d\theta}\frac{d\theta}{dt} = -\dot{\theta}\hat{\mathbf{r}}$.
4) $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\mathbf{k}}, \quad \hat{\mathbf{k}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\theta}}, \quad \hat{\boldsymbol{\theta}} \times \hat{\mathbf{k}} = \hat{\mathbf{r}}$.

The angular momentum vector is expressible as

$$\begin{aligned}
\mathbf{L} &= m(\mathbf{r}\times\mathbf{v}) = m(r\hat{\mathbf{r}})\times(\dot{r}\hat{\mathbf{r}}+r\dot{\theta}\hat{\boldsymbol{\theta}}) \\
&= mr\dot{r}(\hat{\mathbf{r}}\times\hat{\mathbf{r}}) + mr^2\dot{\theta}(\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}}) \\
&= mr^2\dot{\theta}\,\hat{\mathbf{k}}
\end{aligned}$$

So angular momentum points in the $z$-direction as required. The magnitude of the angular momentum is thus

$$L = mr^2\dot{\theta}$$

which is constant in time.

**Exercise:** Problem Sheet 2 Question 7.

Since we are dealing with a central force

$$\mathbf{F} = -V'(r)\hat{\mathbf{r}}$$

the equation of motion is given (see above exercise) by

$$-V'(r)\hat{\mathbf{r}} = m(\ddot{r}-r\dot{\theta}^2)\hat{\mathbf{r}} + m(2\dot{r}\dot{\theta}+r\ddot{\theta})\hat{\boldsymbol{\theta}}.$$

Equating the radial and angular components gives us

$$\ddot{r}-r\dot{\theta}^2 = -\frac{V'(r)}{m}$$

for the radial part and

$$2\dot{r}\dot{\theta}+r\ddot{\theta}=0$$

for the angular equation. Rewriting this last equation, we obtain

$$\frac{d}{dt}(r^2\dot{\theta})=0$$

which is equivalent to stating that the angular momentum

$$L = mr^2\dot{\theta}$$

is constant in time. The motion of the particle is thus governed by the radial equation

$$\ddot{r}-r\dot{\theta}^2 = -\frac{V'(r)}{m}.$$

## 2.15 Inverse square law and planetary motion

According to Newton's law of gravitation a planet of mass $m$ orbiting a star of mass $M$ (at the origin) experiences the central force

$$\begin{aligned} \mathbf{F} &= -\frac{GMm}{r^2}\hat{\mathbf{r}} \\ &= -\frac{K}{r^2}\hat{\mathbf{r}}, \quad \text{K= GMm} \\ &= -\nabla V(r), \end{aligned}$$

where $V(r) = -\frac{K}{r}$ is the potiential energy. Thus the energy of the planet is

$$E = \frac{1}{2}mv^2 - \frac{K}{r}$$

which remains constant in time.

Now introduce the Runge' vector

$$\mathbf{R} = \hat{\mathbf{r}} + \frac{1}{K}(\mathbf{L} \times \mathbf{v}) = \hat{\mathbf{r}} + \frac{L}{K}(\hat{\mathbf{k}} \times \mathbf{v}).$$

**Claim:** $\dot{\mathbf{R}} = 0$, so $\mathbf{R}$ remains constant in time.
**Proof:**

$$\dot{\mathbf{R}} = \frac{d\hat{\mathbf{r}}}{dt} + \frac{L}{K}(\hat{\mathbf{k}} \times \dot{\mathbf{v}}).$$

Using the equation of motion

$$m\dot{\mathbf{v}} = -\frac{K}{r^2}\hat{\mathbf{r}},$$

we obtain

$$\begin{aligned} \dot{\mathbf{R}} &= \dot{\theta}\hat{\boldsymbol{\theta}} - \frac{L}{mr^2}(\hat{\mathbf{k}} \times \hat{\mathbf{r}}) \\ &= \dot{\theta}\hat{\boldsymbol{\theta}} - \frac{L}{mr^2}\hat{\boldsymbol{\theta}} = \frac{1}{mr^2}(mr^2\dot{\theta} - L)\hat{\boldsymbol{\theta}} = \mathbf{0}. \end{aligned}$$

In particular, the length $|\mathbf{R}|$ of $\mathbf{R}$ is constant. Now

$$|\mathbf{R}|^2 = \mathbf{R} \cdot \mathbf{R} = 1 + \frac{L^2}{K^2}|\hat{\mathbf{k}} \times \mathbf{v}|^2 + \frac{2L}{K}\hat{\mathbf{r}} \cdot (\hat{\mathbf{k}} \times \mathbf{v}).$$

But $(\hat{\mathbf{k}} \times \mathbf{v}) = \hat{\mathbf{k}} \times (\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}) = \dot{r}\hat{\boldsymbol{\theta}} - r\dot{\theta}\hat{\mathbf{r}}$ so

$$\begin{aligned} |\mathbf{R}|^2 &= 1 + \frac{L^2}{K^2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{2L}{K}r\dot{\theta} \\ &= 1 + \frac{L^2v^2}{K^2} - \frac{2L^2}{Kmr} \\ &= 1 + \frac{2L^2}{mK^2}(\frac{1}{2}mv^2 - \frac{K}{r}) \end{aligned}$$

from which we can recognize $E$. Therefore

$$|\mathbf{R}| = \left(1 + \frac{2L^2E}{mK^2}\right)^{\frac{1}{2}} = e \geq 0$$

which is called the *eccentricity* of the orbit.

Below we assume that $L \neq 0$ (the case $L = 0$ is trivial - see tutorial sheet).

## 2.16 Planetary orbits

To understand the motion of a planet better, we assume $\mathbf{R}$ points along the $x$-axis.

Then using $\hat{\mathbf{k}} \times \mathbf{v} = \dot{r}\hat{\boldsymbol{\theta}} - r\dot{\theta}\hat{\mathbf{r}}$, show that $r(1 - e\cos\theta) = \frac{L^2}{Km}$.

Solution:
we find

$$\begin{aligned} er\cos\theta &= |\mathbf{R}|\mathbf{r}\cos\theta \\ &= \mathbf{R}\cdot\mathbf{r} \\ &= (\hat{\mathbf{r}} + \frac{L}{K}(\hat{\mathbf{k}} \times \mathbf{v}))\cdot\mathbf{r} \\ &= [\hat{\mathbf{r}} + \frac{L}{K}(\dot{r}\hat{\boldsymbol{\theta}} - r\dot{\theta}\hat{\mathbf{r}})]\cdot\mathbf{r} \\ &= r - \frac{L}{K}r^2\dot{\theta} = r - \frac{L^2}{Km}. \end{aligned}$$

Hence

$$r = \frac{L^2}{Km}\frac{1}{(1 - e\cos\theta)},$$

which is the equation of a conic section with eccentricity $e$ and sun at one focus. We have the following possibilities:

1. $0 \leq e < 1$ - ellipse ($E < 0$) - the case $e = 0$ corresponds to a circle (special case of an ellipse),
2. $e = 1$ - parabola ($E = 0$),
3. $e > 1$ - hyperbola ($E > 0$).

**Notes:**
1) The above also applies to asteroids, comets, etc, orbiting the sun, as well as to satellites (natural or artificial) orbiting the planets.
2) Planets are trapped in sun's gravitational field and so have the energy $E < 0$ corresponding to elliptical orbits. For comets it is possible for $E \geq 0$ corresponding to parabolic or hyperbolic orbits - these are non-periodic comets.

## 2.17 Elliptical orbits $(0 \leq e < 1)$

From the polar equation for $r$ given above, we write

$$r = \frac{\beta}{1 - e\cos\theta}, \quad \beta = \frac{L^2}{mK}.$$

It follows that the maximum value of $r$ occurs at $\theta = 0$:

$$r_{\text{max}} = r_0 = \frac{\beta}{1-e},$$

which is referred to as the aphelion distance from the sun (at the origin).

The minimum value of $r$ occurs at $\theta = \pi$:

$$r_{\text{min}} = r_1 = \frac{\beta}{1+e},$$

which is called the perihelion distance. The length of the semi-major axis is thus

$$a = \frac{1}{2}(r_0 + r_1) = \frac{\beta}{1-e^2}.$$

**Notes:**
1) $r_0 = a(1+e), \quad r_1 = a(1-e)$
2) Centre of ellipse is at $(ea, 0)$.

## 2.18 Cartesian Form

To see that the above orbits are indeed ellipses we note from $r(1 - e\cos\theta) = \beta$ that $r = \beta + er\cos\theta = \beta + ex$. Thus

$$x^2 + y^2 = r^2 = (\beta + ex)^2 = \beta^2 + 2e\beta x + e^2x^2$$

or $(1-e^2)x^2 - 2e\beta x + y^2 = \beta^2$. Rewriting:

$$x^2 - \frac{2e\beta x}{1-e^2} + \frac{y^2}{1-e^2} = \frac{\beta^2}{1-e^2},$$

$$\begin{aligned} \left(x - \frac{e\beta}{1-e^2}\right)^2 + \frac{y^2}{1-e^2} &= \frac{\beta^2}{(1-e^2)} + \frac{\beta^2 e^2}{(1-e^2)^2} \\ &= \frac{\beta^2}{(1-e^2)^2} \end{aligned}$$

or

$$\frac{(x-ea)^2}{a^2} + \frac{y^2}{(1-e^2)a^2} = 1$$

which is the equation of an ellipse centered on $(ea, 0)$ with semi-major axis $a$ and semi-minor axis $b = \sqrt{1-e^2}a$. This explains **Kepler's first law** that the orbits of planets are ellipses with the sun at one focus.

Notes:
1) $1 - e^2 = -\frac{2L^2E}{mK^2} = -\frac{2\beta E}{K}$, therefore $a = \frac{\beta}{1-e^2} = -\frac{K}{2E}$. Then the energy of the planet is given by

$$E = -\frac{K}{2a}.$$

2) The length of the semi-minor axis is thus

$$b = \sqrt{1-e^2}a = \sqrt{\frac{-2L^2E}{mK^2}}a = \sqrt{\frac{L^2}{mKa}}a = \frac{L}{\sqrt{mK}}a^{\frac{1}{2}}.$$

## 2.19 Kepler's third law:

The period $T$ of a revolution of orbit is proportional to $a^{\frac{3}{2}}$.
**Proof:**

$$\begin{aligned} \pi ab &= \text{area of ellipse} \\ &= \int_0^T \frac{dA}{dt} dt \\ &= \int_0^T \frac{L}{2m} dt = \frac{LT}{2m} \end{aligned}$$

Then $T = \frac{2m\pi ab}{L} = \frac{2m\pi a}{L} \frac{L}{\sqrt{mK}} a^{\frac{1}{2}}$ so that

$$T = 2\pi \sqrt{\frac{m}{K}} a^{\frac{3}{2}} = \frac{2\pi}{\sqrt{GM}} a^{\frac{3}{2}}$$

as desired.
**Notes:**
1) All of the above applies to asteroids, captured comets etc. orbiting the sun as well as to satellites orbiting planets.
2) Planets move around the sun in elliptical orbits of small eccentricity - well approximated by circles. Eg. $e_{Earth} = \frac{1}{60}$, $e_{Venus} = \frac{1}{143}$, $e_{Neptune} = \frac{1}{125}$.
3) By contrast periodic comets such as Halley, Enche, etc. have highly elongated elliptical orbits - well approximated by parabolas (particularly near perihelion) eg. $e_{Enche's} \approx .85$.
4) Non-periodic comets have energy $E \geq 0$ and move in parabolic or hyperbolic orbits passing the sun once never to return.