PHYS2100: Hamiltonian mechanics assignment: part 1
Due 5pm Friday 27th October 2006.

1. The Lagrangian for a *relativistic* harmonic oscillator is

$$\mathcal{L}(q,\dot{q}) = -mc^2\sqrt{1-\frac{\dot{q}^2}{c^2}} - \frac{m\omega^2}{2}q^2.$$

(a) Derive an expression for the generalised momentum p.

(b) Hence derive an expression for the Hamiltonian $H(p,q)$ of the system. (N.B. Make sure that you eliminate all terms involving $\dot{q}$.)

(c) Setting $m = \omega = c = 1$, sketch the phase portrait of the system. Hint: consider the limits $p \ll 1$ and $p \gg 1$ to get the qualitative shape.

2. A particle of mass $m = 2$ moves in the potential

$$V(q) = 2q^2 e^{-q^2}.$$

(a) Sketch the potential and the phase portrait.

(b) Find the fixed points, and determine their stability.

(c) If any hyperbolic fixed points exist, find the equation of the separatrices.

3. If k and λ are constants, determine which of the following are canonical transformations:

(a) $Q = q^2/2, \qquad P = p/q.$
(b) $Q = \tan q, \qquad P = (p-k)\cos^2 q.$
(c) $Q = \sin q, \qquad P = (p-k)/\cos q.$
(d) $Q = \sqrt{q}e^{\lambda}\cos p, \qquad P = \sqrt{q}e^{-\lambda}\sin p.$

4. A particle of mass m slides smoothly (without friction) on a plane inclined at an angle α to the horizontal (i.e. when $\alpha = 0$ the plane is perpendicular to gravity). The plane is attached to a vertical wall off which the particle bounces elastically. The coordinate q measures the distance of the particle along the plane from the wall (i.e. it is not the horizontal distance!

(a) Write down the potential function $V(q)$, and sketch a graph labelling all relevant features.

(b) Hence, sketch the phase portrait for the system.

(c) Show that the frequency of motion is

$$\omega(I) = \frac{2}{3}\left(\frac{3}{2}\pi mg\sin\alpha\right)^{2/3}\left(\frac{1}{2mI}\right)^{1/3},$$

where I is the action.

(d) Find the angle variable θ as a function of q, the distance of the particle along the plane measured *from the wall*, for the whole range $0 \Rightarrow 2\pi$. [Hint: you will need to break this calculation into two parts: one for $p > 0$ and the other for $p < 0$.]