Chapter 4

Introduction to chaotic dynamics

We begin this section with a quote from the Introduction of *Chaotic Dynamics of Nonlinear Systems*, by S. Neil Rasband (Wiley, New York, 1990):

"Arguably the most broad based revolution in the worldview of science in the twentieth century will be associated with chaotic dynamics. Yes, I know about Quantum Mechanics and Relativity, and for physicists and philosophers these theories must rank above Chaos for their impact on the way we view the world. My assertion, however, refers to science in general, not just to physics. Leaving improved diagnostic instrumentation aside, it is not clear that Quantum Mechanics or Relativity have had any appreciable effect whatever on medicine, biology, or geology. Yet chaotic dynamics is having an important impact in all these fields, as well as many others, including chemistry and physics.

Surely part of the reason for this broad application is that chaotic dynamics is not something that is part of a specific physical model, limited in its application to one small area of science. But rather chaotic dynamics is a consequence of mathematics itself and hence appears in a broad range of physical systems. Thus, although the mathematical representations of these physical systems can be very different, the often share common properties."

The concept of chaos can be quite hard to define. For our purposes, the best definition is that a dynamical system is chaotic if it displays dynamics that are *highly sensitive to initial conditions.* It is important to note that there is no probability or chance involved: *chaotic dynamics are deterministic.* However, due to the sensitivity to initial condidtions, after a certain time it may appear that the results are totally unrelated to one another.

The main emphasis of this part of the course has been the dynamics of Hamiltonian systems with one degree of freedom. It turns out that such systems do not display chaotic dyanmics, as they are *integrable* — they have as many conserved quantities as degrees of freedom. In this case, the conserved quantity is the energy. If we introduce a time-dependence into the potential $V(q)$ of the system, then the energy is no longer conserved and chaotic dynamics can arise. In particular, we will be looking at the case of the driven pendulum.

However, studying chaos in Hamiltonian dynamics requires the integration of two first order differential equations, which introduces a significant amount of complexity. To begin our introduction to chaotic dynamics, we will look at one-dimensional nonlinear maps which display many of the important features chaos in a very simple system.

4.1 One-dimensional linear maps

One dimensional maps are extremely simple: the are of the form

$$x_{n+1} = f(x_n), \qquad 0 \leq x_n \leq 1, \tag{4.1}$$

i.e. they take a value of x and transform it to another value of x. Linear maps are of the form

$$x_{n+1} = ax_n + b, \tag{4.2}$$

for some constants a, b. Such maps are one-to-one, and they cannot display chaotic behaviour. However, nonlinear maps are many-to-one, and can display chaos. The most commonly studied example of a 1D nonlinear map is the *logisitic map*

$$f(x) = \mu x(1-x), \qquad 0 \leq x \leq 1. \tag{4.3}$$

It is easy to show that for $f(x)$ to be bounded by zero and one, that we must have $\mu \leq 4$. The constant parameter μ dramatically affects the behaviour of the map.

4.1.1 Relation to population dynamics

The logistic map is a discretisation of the *logistic equation*

$$\dot{x} = bx - cx^2, \tag{4.4}$$

which is a very simple model of the dynamics of a biological population x. The populations of insects, birds, fish, and mammals are increased by births and decreased by deaths, the rates of which depend on a very complicated interplay of huge range of influences. The simplest model you can come up with is that the rate equation for the population can be written

$$\dot{x} = [B(x) - D(x)]x, \tag{4.5}$$

where we have assumed that the birth rate $B(x)$ and the death rate $D(x)$ may depend on the current population, but not on space, time, or any other factors. Of course we must have that $B(x), D(x) \geq 0$, and also there is a natural boundary of $x \geq 0$.

The simplest assumption is that the birth and death rate are constants independent of the population, so that

$$\dot{x} = [B - D]x. \tag{4.6}$$

The solution to this equation is exponential growth for $B > D$, or exponential decay for $B < D$.

However, in practice a population in a confined region of space cannot increase without bound forever, as there will be limiting factors such as competition for food and other resources. The next simplest assumption is to suppose that such factors leave the birth rate unchanged, but give a death rate per individual proporation to the population, so that

$$B(x) = b, \qquad D(x) = cx, \tag{4.7}$$

which gives the logisitic equation (4.4). It turns out that some actual populations do follow such an equation closely.

However, the situation is rarely so simple. In particular, often one species preys on another, and so their population equations are coupled together leading to more complicated systems.

Another possible feature is that often individual species have a definite reproductive season, so that the change in population is not represented by a differential equation, but instead by a difference equation or map. By discretising and scaling Eq. (4.4) we can write

$$\begin{aligned} \frac{\Delta x}{x_0} &= b\frac{x}{x_0}\Delta t - c\left(\frac{x}{x_0}\right)^2 \Delta t, & (4.8) \\ \Rightarrow x + \Delta x &= (b\Delta t - 1)x - \frac{c\Delta t}{x_0}x^2. & (4.9) \end{aligned}$$

By identifying $\mu = b\Delta t - 1$ and choosing $x_0 = c\Delta t/\mu$, then we have derived the logistic map

$$x_{n+1} = \mu x_n(1 - x_n). \tag{4.10}$$

This can cause completely new phenomena to appear in the time dependence of the population as compared to the logisitic equation.

4.1.2 Geometrical representation

Let's take a particular example, with $\mu = 1.8$ and $x_0 = 0.7$. Then we find that

$$x_0,\ x_1,\ x_2,\ x_3, \ldots\ =\ 0.7,\ 0.1680,\ 0.1118,\ 0.0795,\ 0.0585,\ \ldots, 0,\ 0, \ldots$$

which eventually converges to the fixed value 0. The mapping can be represented geometrically as in Fig 4.1 and described in the caption. You will have had the opportunity to investigate the logistic map numerically in a computer lab. What you would have found is that for $\mu < 1$, then the mapping eventually converges to 0. Then up until $\mu = 3$, the mapping converges to a finite value of x. For $\mu > 3$, sometimes the mapping settles down to a regular pattern, but other times it doesn't. So it is obvious that some stable fixed points exist in the mapping. The behaviour for particular selections with iteration number n are show in Fig. 4.2. A so called "bifurcation diagram" that attempts to plot the fixed points for the full range of values of μ is shown in Fig. 4.3

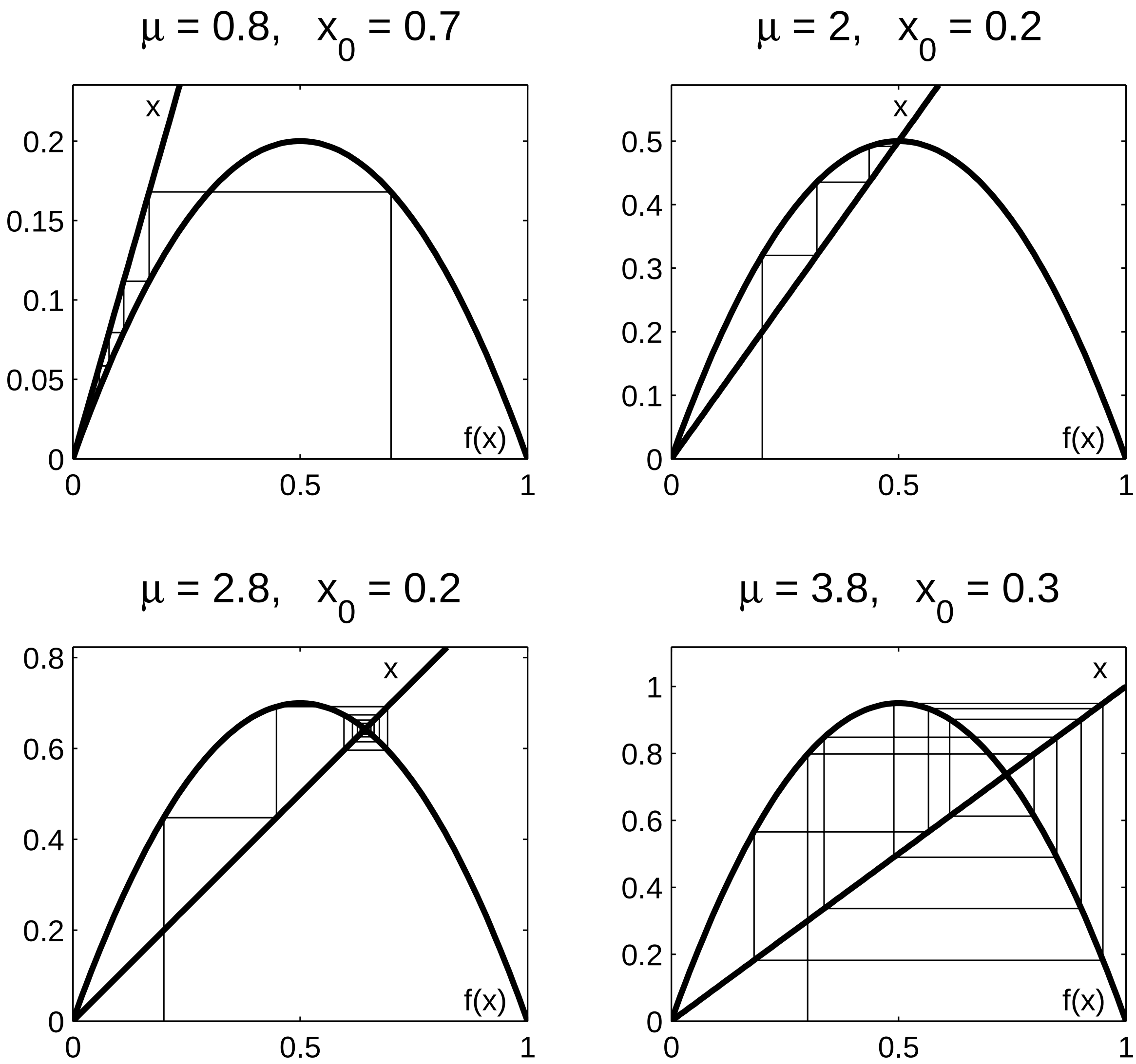

Figure 4.1: Geometical representation of the logistic map for a number of combinations of μ and x_0 for ten iterations. We begin at x_0 on the horizontal axis, and move vertically until we hit $x_1 = f(x_0)$. This is mapped back to an initial condition by moving horizontally we hit the curve x. Then move vertically to give $x_2 = f(x_1)$, and so on.

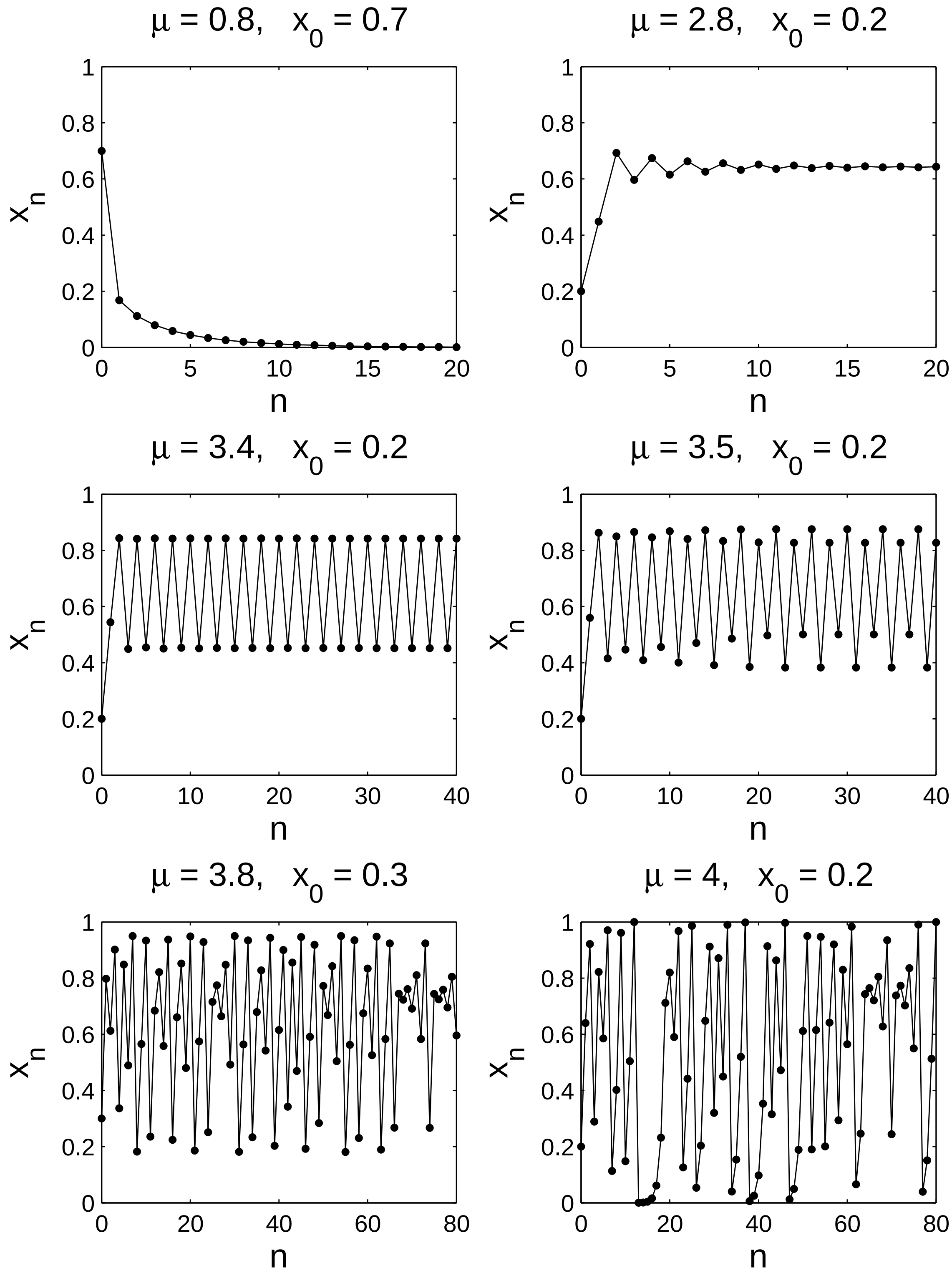

Figure 4.2: Evolution of the logistic map with iteration number n for some particular values of x_0 and μ.

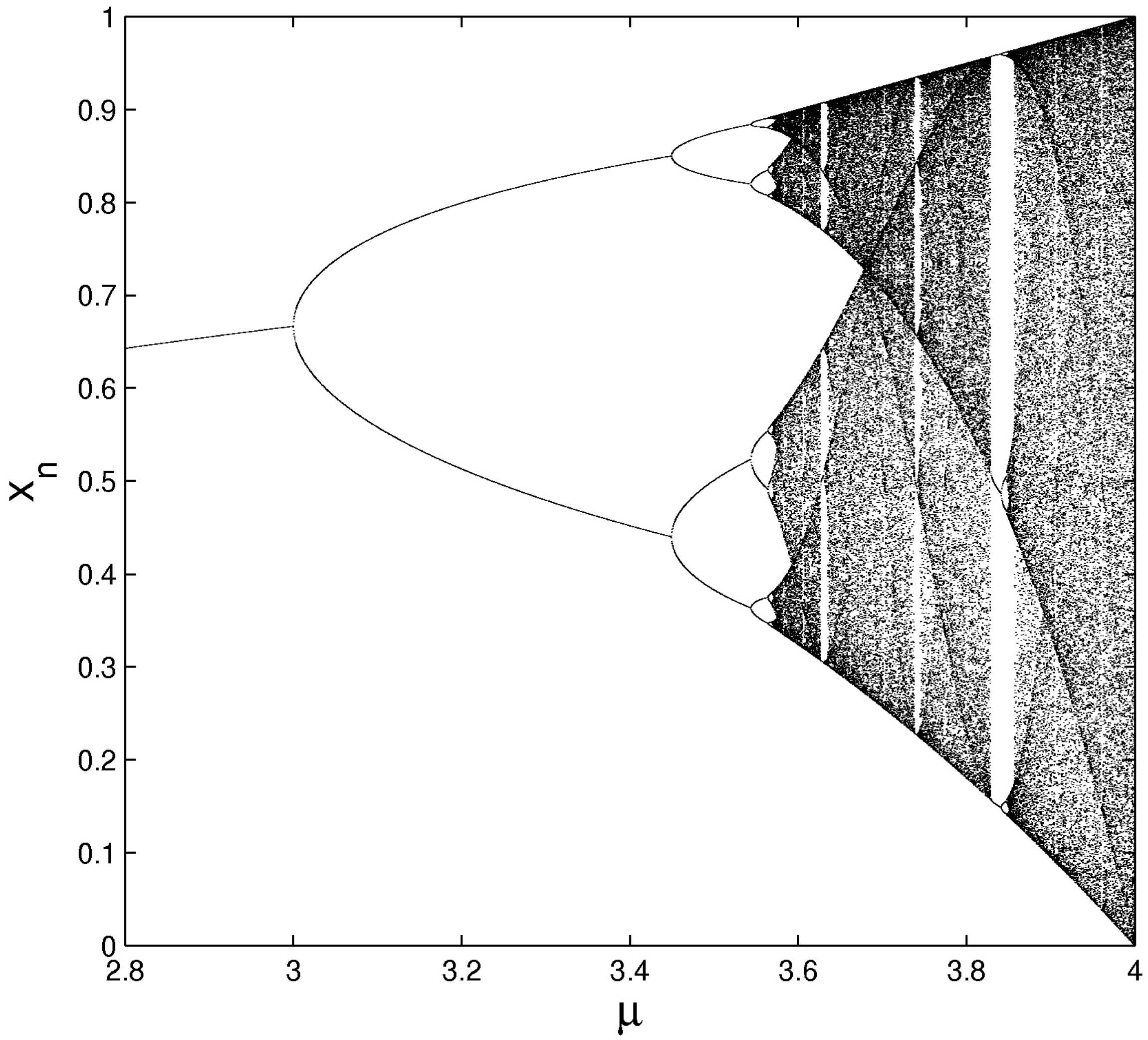

Figure 4.3: Bifurcation diagram for the logistic map for the region $2.8 \leq \mu \leq 4$.

4.1.3 Fixed points

The fixed points of a 1D map x_f are found by setting

$$x_f = f(x_f) \tag{4.11}$$

which for the logistic map is

$$x_f = \mu x_f(1 - x_f), \qquad \Rightarrow \qquad x_f(1 - (\mu - 1)x_f) = 0. \tag{4.12}$$

So the fixed points are

$$x_f = 0, \quad 1 - \frac{1}{\mu}. \tag{4.13}$$

These are just the intercepts of the two curves in Fig. 4.1. As we have $0 \leq x \leq 1$, then the second fixed point can only exist for $\mu \geq 1$.

Lets consider the stability of the fixed points. We define the distance of x_n from the fixed point at x_f by

$$\delta_n = x_n - x_f, \tag{4.14}$$

and we consider this quantity in a small neighborhood of the fixed point x_f. We have

$$\begin{aligned} |\delta_{n+1}| &= |x_{n+1} - x_f| \;=\; |f(x_n) - x_f| \;=\; |f(x_f + \delta_n) - x_f|, && (4.15) \\ &= \Big| f(x_f) + \delta_n f'(x_f) - x_f \Big| \;=\; \Big| f'(x_f) \Big| |\delta_n|, && (4.16) \end{aligned}$$

where we have made a first order Taylor series expansion of $f(x)$ about the fixed point. Clearly $|\delta_{n+1}| < |\delta_n|$ and the fixed point is stable if and only if

$$\left| \frac{df}{dx} \right|_{x=x_f} < 1. \tag{4.17}$$

For the logistic map

$$\frac{df}{dx} = \mu(1 - 2x) \tag{4.18}$$

For the first fixed point $x_f = 0$ then

$$\left| \frac{df}{dx} \right|_{x=0} = |\mu| \tag{4.19}$$

and so this fixed point is stable for $\mu < 1$ as we discovered numerically. For the second fixed point $x_f = 1 - 1/\mu$ and

$$\left| \frac{df}{dx} \right|_{x=1-1/\mu} = |\mu(1 - 2 + 2/\mu)| = |2 - \mu|, \tag{4.20}$$

and so this is stable in the range $1 < \mu < 3$. Thus neither of the fixed points are stable for $3 \leq \mu \leq 4$.

4.1.4 Unstable fixed points

Let us consider the case for $\mu \geq 3$. From Fig. 4.2 we can see that for some values of μ it looks like the map settles down to some sort of periodic behaviour, but for others it seems there is no pattern at all. This is confirmed in the bifurcation diagram Fig. 4.3.

For the regime $3 < \mu < 3.44$ it appears that rather than having a single fixed point, the map oscillates about two values of x, such that $x_{n+2} = x_n$. As the number of steps between identical values of x_n is now two rather than one, the period is said to vhave *doubled*.

As μ is further increased, we can see that around $\mu = 3.45$ then there are four stable values of x, and we have $x_{n+4} = x_n$. So the period has doubled again. As μ is further increased the period doubling mechanism continues until the system is chaotic. The points at which the splits occur in Fig. 4.3 are known as *pitchfork bifurcations*.

The period doubling mechanism is a typical route to chaotic dynamics, and can in general be characterised by certain numbers that in general do not depend on the nature of the map. For example, the ratio of the spacings between consecutive values of μ at the bifurcation points approaches a universal constant called the Feigenbaum constant, that was only discovered in 1980

$$\delta \quad = \quad \lim_{k\to\infty}\left(\frac{\mu_k - \mu_{k-1}}{\mu_{k+1} - \mu_k}\right) \quad = \quad 4.669201609\ldots. \tag{4.21}$$

The values of μ for which bifurcations occur for the logistic map are found to be

$\mu_1 = 3$	$\mu_2 = 3.449490\ldots$
$\mu_3 = 3.544090\ldots$	$\mu_4 = 3.564407\ldots$
$\mu_5 = 3.568759\ldots$	$\mu_6 = 3.569692\ldots$
$\mu_7 = 3.569891\ldots$	$\mu_8 = 3.569934\ldots$

Extrapolating we can see that we have $\mu_\infty = 3.5699456\ldots$.

However, beyond this value it is obvious from Fig. 4.3 that there are still values of μ for which there are stable periodic cycles, the biggest being around $\mu = 3.828427\ldots$ where there is a stable period-3 cycle. Outside these windows, the map appears chaotic.

4.1.5 Lyapunov exponent

It would be of use to be able to define parameters that can be used to characterise chaos. Remember that we defined chaos as being the rapid divergence of nearby points in phase space. This divergence has been parameterised by the *Lyapunov exponent* λ, probably the most popular measure of chaotic behaviour.

Consider a one-dimensional system with initial states x and $x+\epsilon$, where $\epsilon \ll 1$ is a small parameter. After n iterations their divergency $\epsilon(n)$ may be approximated as

$$\epsilon(n) \approx \epsilon e^{n\lambda}. \tag{4.22}$$

Thus the Lyapunov exponent gives the average rate of divergence. If $\lambda < 0$ then separate trajectories converge and the system is not chaotic. However, if $\lambda > 0$ then separate trajectories diverge and the system is chaotic.

For a 1D map

$$x_{n+1} = f(x_n). \tag{4.23}$$

So in going from the initial value x_0 to x_{n+1} the map f has been applied n times

$$x_{n+1} = f^n(x_0). \tag{4.24}$$

The difference in the "state" of the system after n steps is

$$f^n(x_0 + \epsilon) - f^n(x_0) \approx \epsilon e^{n\lambda}, \tag{4.25}$$

and so

$$\lambda \approx \frac{1}{n} \ln \left[\frac{f^n(x_0 + \epsilon) - f^n(x_0)}{\epsilon} \right]. \tag{4.26}$$

In the limit that $\epsilon \to 0$ then we have

$$\lambda = \frac{1}{n} \ln \left[\left. \frac{df^n(x)}{dx} \right|_{x=x_0} \right]. \tag{4.27}$$

Using the chain rule and $\ln ab = \ln a + \ln b$ we can show

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|. \tag{4.28}$$

The Lyapunov exponent for the logistic map is shown in Fig. 4.4.

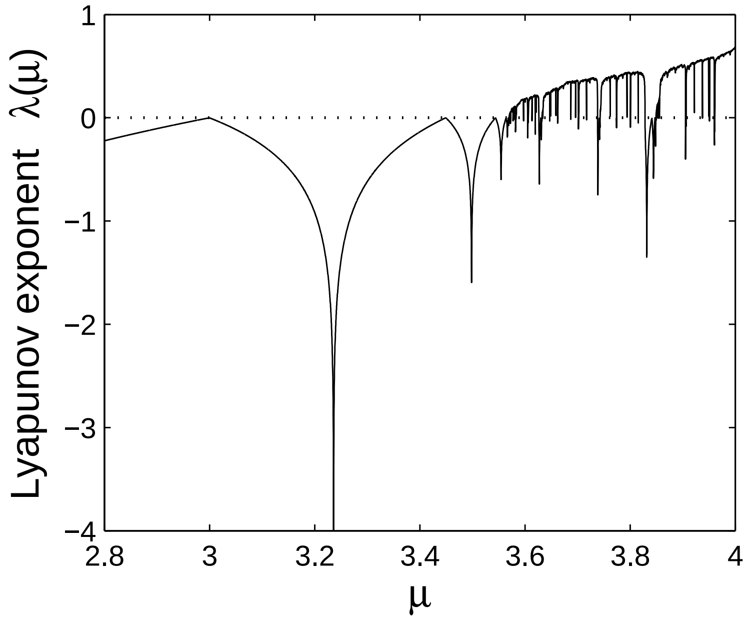

Figure 4.4: Lyapunov exponent for the logistic map for the region $2.8 \leq \mu \leq 4$.