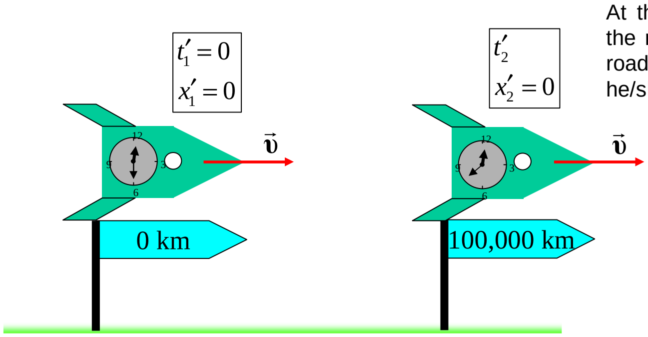
Lecture 3

Space components of the 4-velocity

A rocket has velocity υ in the reference frame where the road is at rest



At this moment the observer on the rocket reads the sign on the road side and gets the "distance" he/she has travelled .

Then the "driver" calculates his/her velocity by dividing the time interval on the clock attached to the rocket by the distance read from the signs standing along the road.

$$\upsilon_{apparent} \equiv \frac{\Delta x_2}{\Delta t_2'} = \frac{\Delta x_2}{\Delta t_2}\frac{\Delta t_2}{\Delta t_2'} = \frac{\upsilon}{\sqrt{1-\upsilon^2/c^2}}$$

Quite obviously, this velocity is larger than υ and is not limited by *c*.

Note that in this case position and time are measured in different frames.

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Transformation of the **3**-velocity

4-velocity is transformed according to the general rules for 4-vectors:

$$V_1' = \gamma\left(V_1 - u/cV_4\right);$$
$$V_2' = V_2;$$
$$V_3' = V_3;$$
$$V_4' = \gamma\left(V_4 - u/cV_1\right)$$

When written using explicit definition of the 4-velicity, these transformations read

$$\gamma(\upsilon')\cdot\upsilon_1' = \gamma(u)\gamma(\upsilon)\cdot(\upsilon_1 - u);$$
$$\gamma(\upsilon')\cdot\upsilon_2' = \gamma(\upsilon)\cdot\upsilon_2;$$
$$\gamma(\upsilon')\cdot\upsilon_3' = \gamma(\upsilon)\cdot\upsilon_3;$$
$$\gamma(\upsilon')c = \gamma(u)\gamma(\upsilon)\cdot\left(c - u/c\upsilon_1\right)$$

Solve the first 3 equations for υ' to get

$$\upsilon_1' = \frac{\gamma(u)\gamma(\upsilon)}{\gamma(\upsilon')}\cdot(\upsilon_1 - u)$$
$$\upsilon_2' = \frac{\gamma(u)\gamma(\upsilon)}{\gamma(\upsilon')}\cdot\frac{\upsilon_2}{\gamma(\upsilon)}; \quad \upsilon_3 = \frac{\gamma(u)\gamma(\upsilon)}{\gamma(\upsilon')}\cdot\frac{\upsilon_3}{\gamma(\upsilon)}$$

The equality $\frac{\gamma(u)\gamma(\upsilon)}{\gamma(\upsilon')} = \frac{1}{1 - u\upsilon_1/c^2}$

can be obtained from the transformation of V_4

Final result:

$$\upsilon_1' = \frac{\upsilon_1 - u}{1 - u\upsilon_1/c^2}$$
$$\upsilon_2' = \frac{1}{1 - u\upsilon_1/c^2}\cdot\frac{\upsilon_2}{\gamma(u)}; \quad \upsilon_3' = \frac{1}{1 - u\upsilon_1/c^2}\cdot\frac{\upsilon_3}{\gamma(u)}$$

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Quotient rule

If for any vector **A** in Minkovski space $A_4Y_4 - A_1Y_1 - A_2Y_2 - A_3Y_3$ is invariant (independent on the choice of the coordinate system), then **Y** is a vector. Proof:

$$-\sum_{n=1}^{4} p_{1n}A_nY_1' - \sum_{n=1}^{4} p_{2n}A_nY_2' - \sum_{n=1}^{4} p_{3n}A_nY_3' + \sum_{n=1}^{4} p_{4n}A_nY_4' = -A_1Y_1 - A_2Y_2 - A_3Y_3 + A_4Y_4$$

for any choice of A. Therefore the following system of equations must be satisfied

$$-p_{11}Y_1' - p_{21}Y_2' - p_{31}Y_3' + p_{41}Y_4' = -Y_1$$
$$-p_{12}Y_1' - p_{22}Y_2' - p_{32}Y_3' + p_{42}Y_4' = -Y_2$$
$$-p_{13}Y_1' - p_{23}Y_2' - p_{33}Y_3' + p_{43}Y_4' = -Y_3$$
$$-p_{14}Y_1' - p_{24}Y_2' - p_{34}Y_3' + p_{44}Y_4' = Y_4$$

Because $\det\left(p_{mn}\right) \neq 0$ this system of linear equation for $\{Y_m\}$ has only one solution.

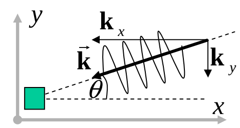
But we know that if $Y_n' = \sum_{m=1}^{4} p_{nm}Y_m$, that is if the $\{Y_m\}$ is transformed as a vector, then the equations are satisfied. Therefore $\{Y_m\}$ must be a vector.

Doppler effect, aberration, and wave velocity transformations

$F_0 \sin\left(\omega t - \vec{\mathbf{k}} \cdot \vec{\mathbf{r}}\right)$

$k = \omega / \upsilon.\ k_x = -k \cos\theta$

υ is the phase velocity of the wave



In an experiment, a recorder (filled box in the Figure) measures oscillating variable *F* related to a wave propagating with phase velocity υ. This experiment can be described using any reference frame. The phase of these oscillations (that is ωt-**kr**) should have the same value in all these frames because its change divided by 2π tells how many maxima have been recorded by the recorder. This outcome of the experiment should not depend on the choice of the reference frame. Therefore the following equality holds

$$\omega t - k_x x - k_y y - k_z z = \omega' t' - k'_x x' - k'_y y' - k'_z z'$$

In other words, $\frac{\omega}{c} c\Delta t - k_x \Delta x - k_y \Delta y - k_z \Delta z$ is invariant.

Therefore (see the quotient rule) $\vec{\mathbf{K}} \equiv \left[\mathbf{k}, \frac{\omega}{c}\right]$

is a 4-wave vector and must be transformed according to the Lorentz transformations

$$k'_x = \gamma\left(k_x - \frac{u}{c^2}\omega\right);\ \ k'_y = k_y;\ \ k'_z = k_z;\ \ \frac{\omega'}{c} = \gamma\left(\frac{\omega}{c} - \frac{u}{c} k_x\right)$$

Note that $\vec{\mathbf{K}} \cdot \vec{\mathbf{K}} = 0$ for EM waves in vacuum

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Lecture 4

Doppler effect

From $\frac{\omega'}{c} = \gamma\left(\frac{\omega}{c} - \frac{u}{c}k_x\right)$ and $k = \omega/\upsilon;\ k_x = -k\cos\theta$

one gets $\omega' = \gamma\left(1 + \frac{u}{\upsilon}\cos\theta\right)\omega$

This is a Doppler frequency shift for any wave. For EM waves in vacuum:

$$\omega' = \gamma\left(1 + \frac{u}{c}\cos\theta\right)\omega$$

The difference between the non relativistic Doppler shift and relativistic one is the factor gamma. Because of this factor, the Doppler shift is also present if θ equals 90 degree. Transverse Doppler shift has been observed experimentally (spectroscopically) for atoms in motion.

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Aberration effect

From $k_x' = \gamma\left(k_x - \frac{u}{c^2}\omega\right);\ k_y' = k_y$ and $k = \omega/\upsilon;\ k_x = -k\cos\theta;\ k_y = -k\sin\theta$

one gets the direction of the wave vector in the primed reference frame.

For any wave, the aberration effect (change of the direction of the phase front propagation)

$$\tan\theta' = \frac{k_y'}{k_x'} = \frac{k_y}{\gamma\left(k_x - \frac{u}{c^2}\omega\right)} = \frac{-\frac{1}{\upsilon}\sin\theta}{\gamma\left(-\frac{1}{\upsilon}\cos\theta - \frac{u}{c^2}\right)} = \frac{\sin\theta}{\gamma\left(\cos\theta + \frac{u\upsilon}{c^2}\right)}$$

For EM waves in vacuum $\tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta + u/c)}$

Other useful relations **for EM waves in vacuum**:

$$\sin\theta' = \frac{-k_y'}{k'} = \frac{kc\sin\theta}{\omega'} = \frac{\sin\theta}{\gamma\left(1 + \frac{u}{c}\cos\theta\right)}$$

$$\cos\theta' = \frac{-k_x'}{k'} = \frac{\cos\theta + \frac{u}{c}}{1 + \frac{u}{c}\cos\theta}$$

$$\tan\frac{\theta'}{2} = \sqrt{\frac{1 - u/c}{1 + u/c}}\tan\frac{\theta}{2}$$

The first two are easy to derive. To derive the last one, you need the identity $\tan\frac{\theta}{2} = \frac{\sin\theta}{1 + \cos\theta}$

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Wave velocity transformation

$\vec{\mathbf{K}}\cdot\vec{\mathbf{K}}$ is invariant. Therefore $\left(\frac{\omega}{c}\right)^2-\left(\frac{\omega}{\upsilon}\right)^2=\left(\frac{\omega'}{c}\right)^2-\left(\frac{\omega'}{\upsilon'}\right)^2$

where we have used the equality $k^2=\omega^2/\upsilon^2$

We substitute the expression for the frequency transformation

$$\left(\frac{\omega}{c}\right)^2-\left(\frac{\omega}{\upsilon}\right)^2=\left(\frac{\gamma\left(1+\frac{u}{\upsilon}\cos\theta\right)\omega}{c}\right)^2-\left(\frac{\gamma\left(1+\frac{u}{\upsilon}\cos\theta\right)\omega}{\upsilon'}\right)^2$$

and solve it for the phase velocity υ'

$$1-\frac{\upsilon^2-c^2}{\gamma^2(\upsilon+u\cos\theta)^2}=\left(\frac{c}{\upsilon'}\right)^2 \qquad \upsilon'=\frac{\gamma(\upsilon+u\cos\theta)c}{\sqrt{\gamma^2(\upsilon+u\cos\theta)^2-\upsilon^2+c^2}}$$

For EM waves in vacuum: $\upsilon'=\frac{\gamma(c+u\cos\theta)c}{\sqrt{\gamma^2(c+u\cos\theta)^2-c^2+c^2}}=c$

4-momentum

Definition of 4-momentum

$$\vec{\mathbf{P}}\equiv m\vec{\mathbf{V}}=\gamma\left[m\vec{\boldsymbol{\upsilon}},mc\right]=\frac{m}{\sqrt{1-\upsilon^2/c^2}}\left[\upsilon_x,\upsilon_y,\upsilon_z,c\right]$$

In these lectures <u>*m* is the rest mass of a particle</u>. Note that in some textbooks the rest mass is labelled as m_0 and $\gamma\, m_0$ is called "relativistic mass".

Axioms of relativistic mechanics

The 4-momentum is the same before and after collision of n particles $\sum_n \vec{\mathbf{P}}_{n,before}=\sum_n \vec{\mathbf{P}}_{n,after}$

By splitting into two parts we get

$$\sum \gamma_n m_n \vec{\boldsymbol{\upsilon}}_n=\text{constant} \quad \text{and} \quad \sum \gamma_n m_n=\text{constant}$$

Relativistic version of 3-momentum conservation

Relativistic version of mass conservation

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Relativistic 3-momentum and total energy

From Newton's physics we know two quantities (one vector and a scalar) which are conserved in any collision. These quantities are momentum and total energy (kinetic energy is conserved only in elastic collisions).

Therefore we identify the vector $\gamma m\vec{\mathbf{v}}$ as relativistic 3-momentum. $\gamma m\vec{\mathbf{v}} \equiv \vec{\mathbf{p}}$

γmc^2 -- can be identified with relativistic total energy. This value looks like energy (mass times square of velocity), it is conserved in all collisions, it looks nice, and it gives correct value for the kinetic energy in a non relativistic limit. Note that when the 3-velocity of the particle is zero, its total energy is mc^2. Therefore, the kinetic energy K is given by

$\gamma mc^2 \equiv E$

$$K = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} - mc^2 \approx mc^2\left(1+\frac{1}{2}\frac{v^2}{c^2}+....\right) - mc^2 \approx \frac{mv^2}{2}$$

Now the 4-momentum can now be also written as $\vec{\mathbf{P}} \equiv \left(\vec{\mathbf{p}}, E/c\right)$

$\vec{\mathbf{P}}^2 \equiv E^2/c^2 - p^2$

Note: For **photons** $E = pc$ and therefore $\vec{\mathbf{P}}^2 = 0$

4-acceleration

Definition of **4**-acceleration: $\vec{\mathbf{A}} \equiv \frac{d\vec{\mathbf{V}}}{d\tau}$

$$\vec{\mathbf{A}} = \frac{d\vec{\mathbf{V}}}{dt}\frac{dt}{d\tau} = \gamma\frac{d\vec{\mathbf{V}}}{dt} = \gamma\frac{d\gamma}{dt}\left[\vec{\mathbf{v}}, c\right] + \gamma^2\left[\vec{\mathbf{a}}, 0\right] = \left[\gamma\frac{d\gamma}{dt}\vec{\mathbf{v}} + \gamma^2\vec{\mathbf{a}}, \gamma\frac{d\gamma}{dt}c\right]$$

Note: $$\frac{d\gamma}{dt} = \frac{d}{dt}\frac{1}{\left(1-v^2/c^2\right)^{1/2}} = \frac{v/c^2}{\left(1-v^2/c^2\right)^{3/2}}\frac{dv}{dt}$$

Examples and simple results:

1. If the length of the 3-velocity is time independent then $\vec{\mathbf{A}} = \left[\gamma^2\vec{\mathbf{a}}, 0\right]$
2. If the 3-velocity is zero then $\vec{\mathbf{A}} = \left[\gamma^2\vec{\mathbf{a}}, 0\right]$
3. The scalar product of 4-acceleration and 4-velocity of the same particle is **always** zero. To prove this note that in the reference frame where the 3-velocity is zero, the 4-velocity is $\left[\vec{\mathbf{0}}, c\right]$

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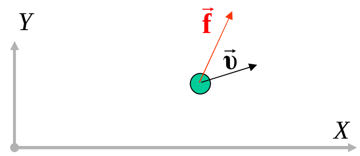
4-force



Definition of **4**-force: $\vec{\mathbf{F}} \equiv \frac{d}{d\tau}\vec{\mathbf{P}}$

Relation between force and acceleration: $\vec{\mathbf{F}} \equiv \frac{d}{d\tau}\vec{\mathbf{P}} = \frac{d}{d\tau}\left(m\vec{\mathbf{V}}\right) = \frac{dm}{d\tau}\vec{\mathbf{V}} + m\frac{d\vec{\mathbf{V}}}{d\tau} = \frac{dm}{d\tau}\vec{\mathbf{V}} + m\vec{\mathbf{A}}$

3-force and **4**-force $\vec{\mathbf{F}} \equiv \frac{d}{d\tau}\vec{\mathbf{P}} = \frac{d}{dt}\left[\vec{\mathbf{p}}, \frac{E}{c}\right]\frac{dt}{d\tau} = \gamma(\upsilon)\left[\vec{\mathbf{f}}, \frac{1}{c}\frac{dE}{dt}\right]$

Note: **p** is a relativistic 3-momentum $\vec{\mathbf{p}} = \gamma m\vec{\boldsymbol{\upsilon}}$ and represents first 3 coordinates of $\vec{\mathbf{P}}$, i e $\left[P_x, P_y, P_z\right]$

3-force is a time derivative of the relativistic 3-momentum $\vec{\mathbf{f}} \equiv d\vec{\mathbf{p}}/dt$. Multiply it by γ to get $\left[F_x, F_y, F_z\right]$

$\boldsymbol{\upsilon}$ is an ordinary 3-velocity $\vec{\boldsymbol{\upsilon}} \equiv \left[dx/dt, dy/dt, dz/dt\right]$. Multiply it by γ to get $\left[V_x, V_y, V_z\right]$

A bit confusing, indeed. Useful equalities related to the 4-force are derived below.

$$\vec{\mathbf{F}}\cdot\vec{\mathbf{V}} = \frac{dm}{d\tau}c^2 + m\vec{\mathbf{A}}\cdot\vec{\mathbf{V}} = \frac{dm}{d\tau}c^2$$

On the other hand $\vec{\mathbf{F}}\cdot\vec{\mathbf{V}} = \gamma^2\frac{dE}{dt} - \gamma^2\vec{\mathbf{f}}\cdot\vec{\boldsymbol{\upsilon}}$

Therefore $\frac{dm}{d\tau}c^2 = \gamma^2\frac{dE}{dt} - \gamma^2\vec{\mathbf{f}}\cdot\vec{\boldsymbol{\upsilon}}$ and

if $\frac{dm}{d\tau} = 0$ then $\frac{dE}{dt} = \vec{\mathbf{f}}\cdot\vec{\boldsymbol{\upsilon}}$

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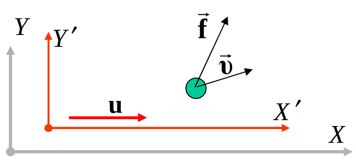
Transformation of **3**-force

Transformation of **4**-force



According to the standard Lorentz transformations

$$F_1' = \gamma\left(F_1 - \frac{u}{c}F_4\right)$$
$$F_2' = F_2; \quad F_3' = F_3$$
$$F_4' = \gamma\left(F_4 - \frac{u}{c}F_1\right)$$

$$\vec{\mathbf{F}} = \gamma(\upsilon)\left[\vec{\mathbf{f}}, \frac{1}{c}\frac{dE}{dt}\right]$$

We express the 4-force components using the 3-force components and the power: $\gamma(\upsilon')\, f_1' = \gamma(u)\left(\gamma(\upsilon)\, f_1 - \frac{u}{c^2}\gamma(\upsilon)\frac{dE}{dt}\right)$

$$\gamma(\upsilon')\, f_2' = \gamma(\upsilon)\, f_2; \quad \gamma(\upsilon')\, f_3' = \gamma(\upsilon)\, f_3$$

$$\gamma(\upsilon')\frac{1}{c}\frac{dE'}{dt'} = \gamma(u)\left(\gamma(\upsilon)\frac{1}{c}\frac{dE}{dt} - \frac{u}{c}\gamma(\upsilon)\, f_1\right)$$

After obvious algebra ane gets:

$$f_1' = \frac{\gamma(u)\gamma(\upsilon)}{\gamma(\upsilon')}\left(f_1 - \frac{u}{c^2}\frac{dE}{dt}\right); \quad f_2' = \frac{\gamma(\upsilon)}{\gamma(\upsilon')}f_2; \quad f_3' = \frac{\gamma(\upsilon)}{\gamma(\upsilon')}f_3$$

$$\frac{dE'}{dt'} = \frac{\gamma(u)\gamma(\upsilon)}{\gamma(\upsilon')}\left(\frac{dE}{dt} - uf_1\right)$$

This can be simplified using the identity (see velocity transformations) $\frac{\gamma(\upsilon')}{\gamma(\upsilon)} = \gamma(u)\left(1 - \frac{\upsilon_x u}{c^2}\right)$

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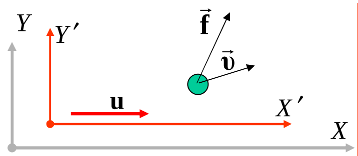
Transformation of 3-force (cont)



After simplification:

$$f_1' = \frac{1}{1-\upsilon_1 u/c^2}\left(f_1 - \frac{u}{c^2}\frac{dE}{dt}\right)$$

$$f_2' = \frac{1}{\gamma(u)\left(1-\upsilon_1 u/c^2\right)} f_2;$$

$$f_3' = \frac{1}{\gamma(u)\left(1-\upsilon_1 u/c^2\right)} f_3$$

$$\frac{dE'}{dt'} = \frac{1}{1-\upsilon_1 u/c^2}\left(\frac{dE}{dt} - u f_1\right)$$

The transformations of the 3-force are similar to the transformation of the 3-velocity. This is not surprising because there is a clear analogy in the expressions for a 4-force and a 4-velocity.

$$\vec{\mathbf{F}} = \gamma(\upsilon)\left[\vec{\mathbf{f}}, c\left(\frac{1}{c^2}\frac{dE}{dt}\right)\right]$$

$$\vec{\mathbf{V}} = \gamma(\upsilon)\left[\vec{\boldsymbol{\upsilon}}, c\right]$$

Two special cases:

1. A rest mass preserving force

If $\frac{dm}{d\tau} = 0$ then $f_1' = \frac{1}{1-\upsilon_1 u/c^2}\left(f_1 - \frac{u\vec{\mathbf{f}}\cdot\vec{\boldsymbol{\upsilon}}}{c^2}\right)$

See derivation of $\frac{dE}{dt} = \vec{\mathbf{f}}\cdot\vec{\boldsymbol{\upsilon}}$ on the 4-force page

2. If $\vec{\mathbf{f}} \equiv \left(f_1, 0, 0\right)$, $\vec{\mathbf{f}} \parallel \vec{\boldsymbol{\upsilon}}$, and $dm/d\tau = 0$ then $f_1' = f_1$

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Transformation of magnetic and electrical fields

The Lorentz force can be used to define the electrical and magnetic fields

The Lorentz 3-force reads $\mathbf{f} = q\boldsymbol{\upsilon}\times\mathbf{b} + q\mathbf{e}$

Note: The form of this equation depends on the units used. For example, in the Gaussian units the Lorentz force is $\mathbf{f} = q\boldsymbol{\upsilon}\times\mathbf{b}/c + q\mathbf{e}$

For briefness we set $q = 1$

Because $\boldsymbol{\upsilon}\times\mathbf{B} = \mathbf{i}\left(\upsilon_2 b_3 - \upsilon_3 b_2\right) + \mathbf{j}\left(\upsilon_3 b_1 - \upsilon_1 b_3\right) + \mathbf{k}\left(\upsilon_1 b_2 - \upsilon_2 b_1\right)$

the components of the 3-force read

$$f_1 = \upsilon_2 b_3 - \upsilon_3 b_2 + e_1$$
$$f_2 = \upsilon_3 b_1 - \upsilon_1 b_3 + e_2$$
$$f_3 = \upsilon_1 b_2 - \upsilon_2 b_1 + e_3$$

In a primed reference frame these components are

$$f_1' = \upsilon_2' b_3' - \upsilon_3' b_2' + e_1'$$
$$f_2' = \upsilon_3' b_1' - \upsilon_1' b_3' + e_2'$$
$$f_3' = \upsilon_1' b_2' - \upsilon_2' b_1' + e_3'$$

Transformation of magnetic and electrical fields

We use the velocity transformations

$$\upsilon_1' = \frac{\upsilon_1 - u}{1 - u\upsilon_1/c^2};\ \upsilon_2' = \frac{\upsilon_2}{\gamma(u)\left(1 - u\upsilon_1/c^2\right)};\ \upsilon_3' = \frac{\upsilon_3}{\gamma(u)\left(1 - u\upsilon_1/c^2\right)}$$

to express the transformed force in terms of non transformed velocity. For example, for the first component of the transformed force we get

$$f_1' = \frac{\upsilon_2}{\gamma(u)\left(1 - u\upsilon_1/c^2\right)} b_3' - \frac{\upsilon_3}{\gamma(u)\left(1 - u\upsilon_1/c^2\right)} b_2' + e_1'$$

On the other hand, we can use general relativistic force transformation for the first component of the force.

$$f_1' = \frac{f_1 - u\mathbf{f} \cdot \boldsymbol{\upsilon}/c^2}{1 - u\upsilon_1/c^2}$$

Transformation of magnetic and electrical fields

We substitute the expressions for f in the not primed reference frame and get

$$f_1' = \frac{\upsilon_2 b_3 - \upsilon_3 b_2 + e_1 - u\left(e_1\upsilon_1 + e_2\upsilon_2 + e_3\upsilon_3\right)/c^2}{1 - u\upsilon_1/c^2} =$$

$$= \frac{\upsilon_2 b_3 - \upsilon_3 b_2 + e_1 - e_1 u\upsilon_1/c^2 - e_2 u\upsilon_2/c^2 - e_3 u\upsilon_3/c^2}{1 - u\upsilon_1/c^2} =$$

$$= \frac{b_3 - e_2 u/c^2}{1 - u\upsilon_1/c^2}\upsilon_2 - \frac{b_2 + e_3 u/c^2}{1 - u\upsilon_1/c^2}\upsilon_3 + e_1$$

Note that

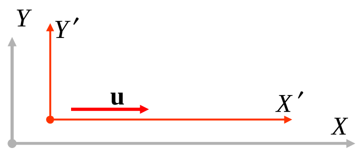
$$\mathbf{f} \cdot \boldsymbol{\upsilon} =$$
$$= (\boldsymbol{\upsilon} \times \mathbf{b}) \cdot \boldsymbol{\upsilon} + \mathbf{e} \cdot \boldsymbol{\upsilon} =$$
$$= \mathbf{e} \cdot \boldsymbol{\upsilon}$$

This can be compared to the expression $f_1' = \frac{b_3'}{\gamma\left(1 - u\upsilon_1/c^2\right)}\upsilon_2 - \frac{b_2'}{\gamma\left(1 - u\upsilon_1/c^2\right)}\upsilon_3 + e_1'$

derived using the velocity transformations. Such comparison gives the following relations

$$b_3' = \gamma\left(b_3 - e_2 u/c^2\right) \qquad b_2' = \gamma\left(b_2 + e_3 u/c^2\right) \qquad e_1' = e_1$$

Transformation of magnetic and electrical fields (summary)



$$e'_x = e_x \qquad b'_x = b_x$$
$$e'_y = \gamma\left(e_y - ub_z\right) \qquad b'_y = \gamma\left(b_y + e_z u/c^2\right)$$
$$e'_z = \gamma\left(e_z + ub_y\right) \qquad b'_z = \gamma\left(b_z - e_y u/c^2\right)$$

The Maxwell equations in vacuum (SI units)

$$\operatorname{div}\vec{\mathbf{b}} = 0; \qquad \operatorname{curl}\vec{\mathbf{b}} = -\mu_0\vec{\mathbf{j}} + \varepsilon_0\mu_0\frac{\partial\vec{\mathbf{e}}}{\partial t}$$
$$\operatorname{div}\vec{\mathbf{e}} = \rho/\varepsilon_0; \quad \operatorname{curl}\vec{\mathbf{e}} = -\frac{\partial\vec{\mathbf{b}}}{\partial t}$$

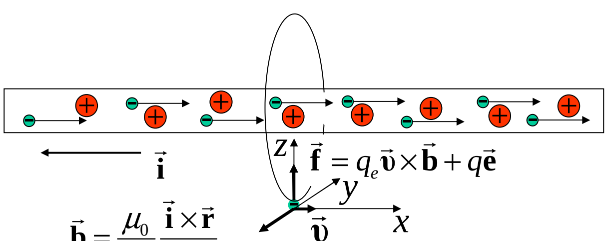
Conversion to Gaussian units

$$\vec{\mathbf{b}}_{SI} = \vec{\mathbf{b}}_G\left(\mu_0/4\pi\right)^{1/2} \qquad \vec{\mathbf{e}}_{SI} = \left(4\pi\varepsilon_0\right)^{-1/2}\vec{\mathbf{e}}_G$$
$$\rho_{SI} = \left(4\pi\varepsilon_0\right)^{1/2}\rho_G; \quad \varepsilon_0\mu_0 = 1/c^2$$

stay valid if the Lorentz transformations of space-time are used, the e and b field is changed as given above, and the current density **j** and the charge density ρ are changed as components of a 4-current density $\vec{\mathbf{J}} \equiv \rho_0\vec{\mathbf{V}} = \rho_0\gamma[\vec{\mathbf{v}}, c] \equiv \left[\vec{\mathbf{j}}, c\rho\right]$ where ρ_0 is the proper charge density.

One can also introduce an electromagnetic field tensor (a generalization of a 4-vector) and write the Maxwell equations in a 4-tensor form).

Lorentz force revisited



$$\vec{\mathbf{b}} = \frac{\mu_0}{2\pi}\frac{\vec{\mathbf{i}}\times\vec{\mathbf{r}}}{r^2}$$

$$f_1 \equiv f_x = 0$$
$$f_2 \equiv f_y = 0$$
$$f_3 \equiv f_z \neq 0$$
$$\vec{\mathbf{u}} = \vec{\mathbf{v}}$$
$$\vec{\mathbf{v}}\cdot\vec{\mathbf{f}} = 0$$

$$f'_1 = \frac{1}{1 - \upsilon u/c^2}\left(f_1 - \frac{\upsilon}{c^2}\vec{\mathbf{f}}\cdot\vec{\mathbf{v}}\right) = 0$$
$$f'_2 = 0$$
$$f'_3 = \frac{\sqrt{1-u^2/c^2}}{1-u^2/c^2}f_3 = \frac{1}{\sqrt{1-u^2/c^2}}f_3$$

This result can be obtained by direct consideration of relativistic kinematics (see Problem 1, Assignment)

$$e'_3 = \frac{\upsilon}{\sqrt{1-\upsilon^2/c^2}}b_y \text{ and } b'_y = \frac{1}{\sqrt{1-\upsilon^2/c^2}}b_y$$

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