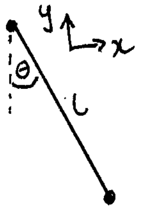
3 Many particle systems: Constraints and generalised coordinates

If we have a system of n particles, $3n$ coordinates or variables are needed to specify their positions. If there are m algebraic constraint equations relating the coordinates, in principle we can eliminate m variables leaving a system depending on $3n-m$ **generalised coordinates** $q_i (1 \leq i \leq 3n-m)$. Such a system is said to have $3n-m$ degrees of freedom.

Examples:

1) Simple Pendulum. Here we have 2 coordinates x, y and one constraint equation $x^2 + y^2 = l^2$. Therefore there is only one generalised coordinate required - in this case θ.

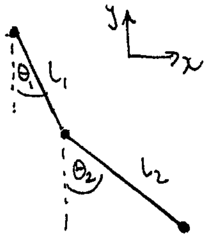


$$\underset{\sim}{r} = l \sin\theta\, \underset{\sim}{\hat{i}} - l\cos\theta\, \underset{\sim}{\hat{j}}$$

2) Double Pendulum. Here we have 4 coordinates x_1, y_1, x_2, y_2 and two constraint equations

$$x_1^2 + y_1^2 = l_1^2, \quad (x_2 - x_1)^2 + (y_2 - y_1)^2 = l_2^2$$

Hence we can transform to two new generalised coordinates θ_1, θ_2.



$$\underset{\sim}{r}_1 = l_1 \sin\theta_1\, \underset{\sim}{\hat{i}} - l_1 \cos\theta_1\, \underset{\sim}{\hat{j}}$$

$$\underset{\sim}{r}_2 = (l_1 \sin\theta_1 + l_2 \sin\theta_2)\, \underset{\sim}{\hat{i}} - (l_1\cos\theta_1 + l_2 \cos\theta_2)\, \underset{\sim}{\hat{j}}$$

We can solve the system solely in terms of θ_1, θ_2.

Notes: For a conservative system with m degrees of freedom and generalised coordinates $q_i \ \ (1 \leq i \leq m)$ (including the single and double pendulums), the K.E. and P.E. can be written as

$$T = T(q_i, \dot{q}_i, t), \quad V = V(q_i, t)$$

respectively. That is, V is not velocity dependent (the Lorentz force will not be considered here).

3.1 Generalised Forces

Consider a system of n particles with m degrees of freedom and generalised coordinates $q_i \ \ (1 \leq i \leq m)$. The rate at which work is done is

$$\frac{dW}{dt} = \frac{dT}{dt} = \sum_{i=1}^{n} \mathbf{F}_i \cdot \mathbf{v}_i = \sum_{i=1}^{n} \mathbf{F}_i \cdot \dot{\mathbf{r}}_i$$

where $\mathbf{F}_i$ is the vector force exerted on the ith particle, and $\mathbf{r}_i, \mathbf{v}_i = \dot{\mathbf{r}}_i$, are the position and velocity vectors of the ith particle. Suppose $\mathbf{r}_i = \mathbf{r}_i(q_j, t)$. Then

$$\dot{\mathbf{r}}_i = \sum_{j=1}^{m} \left(\frac{\partial \mathbf{r}_i}{\partial q_j} \dot{q}_j \right) + \frac{\partial \mathbf{r}_i}{\partial t}.$$

Hence the rate at which work is done is now given by:

$$\begin{aligned} \frac{dW}{dt} = \frac{dT}{dt} &= \sum_{i=1}^{n} \mathbf{F}_i \cdot \dot{\mathbf{r}}_i \\ &= \sum_{j=1}^{m} Q_j \dot{q}_j + \sum_{i=1}^{n} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial t} \end{aligned}$$

where

$$Q_j = \sum_{i=1}^{n} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j},$$

which we call a **generalised force.**

3.2 Conservative Systems

A system is called **conservative** if there exists a function $V = V(q_i, t)$ called the **P.E.** such that:

$$Q_j = -\frac{\partial V}{\partial q_j}$$

and

$$\sum_{i=1}^{n} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial t} = -\frac{\partial V}{\partial t}.$$

Then

$$\begin{aligned} \frac{dT}{dt} &= -\sum_{j=1}^{m} \frac{\partial V}{\partial q_j} \dot{q}_j - \frac{\partial V}{\partial t} \\ &= -\frac{dV}{dt} \end{aligned}$$

So $T + V = E$, where T is our usual kinetic energy ($T = \sum_{i=1}^{n} \frac{1}{2} m_i v_i^2$).

Note that we usually say that the potential $V = V(q_i)$ has no explicit dependence on time and so $\frac{\partial V}{\partial t} = 0$. Throughout, unless otherwise stated, we make this assumption. Also in most applications (eg. time-independent constraints)

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, ..., q_m) \quad \Rightarrow \quad \frac{\partial \mathbf{r}_i}{\partial t} = 0,$$

although this is not always the case.

3.3 Equilibrium in conservative systems

For a conservative system, the generalised forces are

$$Q_j = -\frac{\partial V}{\partial q_j}.$$

If all the generalised forces in a system are zero, then the system is said to be in **equilibrium**.

If we displace a conservative system from equilibrium by an arbitrarily small amount, say $(\delta q_1, \ldots, \delta q_m)$, then the change of work is zero.
Proof:

$$\begin{aligned} \delta W = \delta T &= -\delta V \\ &= -\sum_j \frac{\partial V}{\partial q_j}\delta q_j \\ &= -\sum_j 0 \cdot \delta q_j \quad \text{(because our gen. forces are zero)} \\ &= 0. \end{aligned}$$

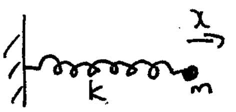
That is, there is no work done in a virtual displacement from equilibrium.

The equilibrium is said to be **stable** (respectively **unstable**) if we have a local minimum (respectively maximum) at equilibrium. It is a minimum (respectively maximum) if the matrix

$$-\frac{\partial Q_j}{\partial q_i} = \frac{\partial^2 V}{\partial q_i \partial q_j} = \frac{\partial^2 V}{\partial q_j \partial q_i} = -\frac{\partial Q_i}{\partial q_j}$$

is positive (respectively negative) definite for all j. That is all the eigenvalues are positive (respectively negative).

Example 9. $m\ddot{x} = -kx$, $Q \equiv F = -kx$, therefore $x = 0$ is an equilibrium position. So

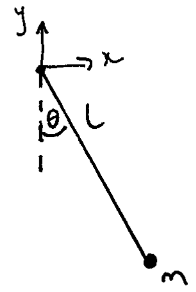


$$-\frac{\partial Q}{\partial x} = -\frac{dF}{dx} = k > 0 \Rightarrow \text{stable equilibrium.}$$

Example 10. Simple pendulum. We have $\mathbf{r} = l\hat{\mathbf{r}}$, $\hat{\mathbf{r}} = \sin\theta\,\hat{\mathbf{i}} - \cos\theta\,\hat{\mathbf{j}}$. Also $\mathbf{v} = \dot{\mathbf{r}} = l\dot{\theta}\hat{\boldsymbol{\theta}}$. Now θ is our generalised coordinate, and we know $\frac{d\hat{\mathbf{r}}}{d\theta} = \hat{\boldsymbol{\theta}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}$. Now

$$\mathbf{F} = -T\hat{\mathbf{r}} - mg\,\hat{\mathbf{j}}$$

where the first term represents the tension force and the second term is gravity. Then



$$\begin{aligned} Q &= \mathbf{F} \cdot \frac{d\mathbf{r}}{d\theta} = l\mathbf{F} \cdot \hat{\boldsymbol{\theta}} \\ &= -mgl\sin\theta \\ &= -\frac{dV}{d\theta}, \quad V = -mgl\cos\theta, \end{aligned}$$

so the force is conservative with P.E. V. Therefore $Q = 0$ at $\theta = 0, \pi$ giving the equilibrium positions. To check stability,

$$\frac{d^2V}{d\theta^2} = mgl\cos\theta = \begin{cases} > 0, & \theta = 0 \text{ stable,} \\ < 0, & \theta = \pi \text{ unstable.} \end{cases}$$

Example 11. Double pendulum.

The system is conservative with potential

$$\begin{aligned} V &= -m_1 g l_1 \cos\theta_1 - m_2 g(l_2\cos\theta_2 + l_1\cos\theta_1) \\ &= -(m_1+m_2)gl_1\cos\theta_1 - m_2 g l_2 \cos\theta_2. \end{aligned}$$

Therefore,

$$-Q_1 = \frac{\partial V}{\partial \theta_1} = (m_1+m_2)gl_1\sin\theta_1,$$

and

$$-Q_2 = \frac{\partial V}{\partial \theta_2} = m_2 g l_2 \sin\theta_2.$$

Hence the system is in equilibrium when $\theta_1 = 0, \pi$ and $\theta_2 = 0, \pi$. Now

$$\frac{\partial^2 V}{\partial \theta_1^2} = (m_1+m_2)gl_1\cos\theta_1, \quad \frac{\partial^2 V}{\partial \theta_2^2} = m_2 g l_2 \cos\theta_2.$$

Also

$$\frac{\partial^2 V}{\partial\theta_1\partial\theta_2} = \frac{\partial^2 V}{\partial\theta_2\partial\theta_1} = 0$$

so

$$\left(\frac{\partial^2 V}{\partial\theta_i\partial\theta_j}\right)_{\theta_1=\theta_2=0} = \begin{pmatrix} (m_1+m_2)gl_1\cos\theta_1 & 0 \\ 0 & m_2 g l_2\cos\theta_2 \end{pmatrix}$$

When $\theta_1 = \theta_2 = 0$ this matrix is positive definite, so we have a stable equilibrium.
When $\theta_1 = \theta_2 = \pi$ this matrix is negative definite, so we have an unstable equilibrium.

Note: When $\theta_1 = 0$ and $\theta_2 = \pi$ the equilibrium is neither stable nor unstable, and similarly for $\theta_1 = \pi$ and $\theta_2 = 0$.