4 Lagrange's Method

Recall the definition of a generalised force

$$Q_j = \sum_{i=1}^{n} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = \sum_i m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}.$$

In dynmaics, the K.E. T of a system is a function of $q_i, \dot{q}_i$ and possibly t: $T = T(q_i \dot{q}_i, t)$. Here we show that the generalised force is related to the kinetic energy of a system with coordinates $q_i, \quad 1 \leq j \leq n$ by:

$$Q_i = \frac{d}{dt}\left\{\frac{\partial T}{\partial \dot{q}_i}\right\} - \frac{\partial T}{\partial q_i}.$$

Suppose the kth particle has position $\mathbf{r}_k = \mathbf{r}(q_1, q_2, \ldots, q_n, t)$. Then

$$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = \sum_j \frac{\partial \mathbf{r}}{\partial q_j}\dot{q}_j + \frac{\partial \mathbf{r}}{\partial t},$$

so

$$\frac{\partial \dot{\mathbf{r}}}{\partial \dot{q}_i} = \frac{\partial \mathbf{r}}{\partial q_i}$$

and

$$\frac{\partial}{\partial q_i}\frac{d\mathbf{r}}{dt} = \frac{\partial \dot{\mathbf{r}}}{\partial q_i} = \sum_j \frac{\partial^2 \mathbf{r}}{\partial q_j \partial q_i}\dot{q}_j + \frac{\partial^2 \mathbf{r}}{\partial q_i \partial t}. \tag{4}$$

Next consider the derivative with respect to time

$$\frac{d}{dt}\left(\frac{\partial \mathbf{r}}{\partial q_i}\right) = \sum_j \frac{\partial^2 \mathbf{r}}{\partial q_i \partial q_j}\dot{q}_j + \frac{\partial^2 \mathbf{r}}{\partial t \partial q_i}. \tag{5}$$

Equating (4) and (5), we see that

$$\frac{\partial \dot{\mathbf{r}}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial \mathbf{r}}{\partial q_i}\right).$$

Now,

$$\begin{aligned} Q_i &= \left\{\sum_j m_j \ddot{\mathbf{r}}_j \cdot \frac{\partial \mathbf{r}_j}{\partial q_i}\right\} \\ &= \frac{d}{dt}\left\{\sum_j m_j \dot{\mathbf{r}}_j \cdot \frac{\partial \mathbf{r}_j}{\partial q_i}\right\} - \sum_j m_j \dot{\mathbf{r}}_j \cdot \frac{d}{dt}\left(\frac{\partial \mathbf{r}_j}{\partial q_i}\right). \end{aligned}$$

For the first sum we have

$$\begin{aligned}
\sum_j m_j \dot{\mathbf{r}}_j \cdot \frac{\partial \mathbf{r}_j}{\partial q_i} &= \sum_j m_j \dot{\mathbf{r}}_j \cdot \frac{\partial \dot{\mathbf{r}}_j}{\partial \dot{q}_i} \\
&= \sum_j \frac{\partial}{\partial \dot{q}_i} (\frac{1}{2} m_j \dot{\mathbf{r}}_j \cdot \dot{\mathbf{r}}_j) \\
&= \frac{\partial}{\partial \dot{q}_i} \left(\sum_j \frac{1}{2} m_j \dot{\mathbf{r}}_j^2 \right) \\
&= \frac{\partial T}{\partial \dot{q}_i}.
\end{aligned}$$

The second summation may be written as

$$\begin{aligned}
\sum_j m_j \dot{\mathbf{r}}_j \cdot \frac{d}{dt} \left(\frac{\partial \mathbf{r}_j}{\partial q_i} \right) &= \sum_j m_j \dot{\mathbf{r}}_j \cdot \frac{\partial \dot{\mathbf{r}}_j}{\partial q_i} \\
&= \sum_j \frac{\partial}{\partial q_i} (\frac{1}{2} m_j \dot{\mathbf{r}}_j \cdot \dot{\mathbf{r}}_j) \\
&= \frac{\partial T}{\partial q_i}.
\end{aligned}$$

Substituting these equations into the above equation for Q_i gives

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} = Q_i.$$

4.1 Lagrange's Equation

Given a conservative system with generalised coordinates $q_i, 1 \le i \le m$, kinetic energy $T = T(q_i, \dot{q}_i, t)$ and potential energy $V = V(q_i, t)$, we can write

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0, \quad 1 \le i \le m \quad \text{(Lagrange's Equation)}$$

where $L = T - V$. L is known as the **Lagrangian** of the system.

Proof: We know from the above that

$$Q_i = \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_i}.$$

As the system is conservative, we also know

$$Q_i = -\frac{\partial V}{\partial q_i}.$$

Moreover, note that the potential energy V is independent of $\dot{q}_i$, so

$$\frac{\partial V}{\partial \dot{q}_i} = 0.$$

Combining the first two equations we get:

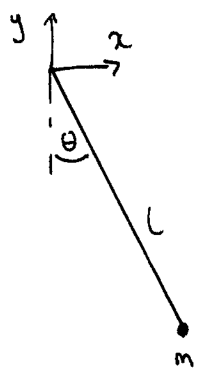
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = -\frac{\partial V}{\partial q_i}.$$

Rearranging, we find

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i} - 0\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} &= 0 \\
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial V}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} &= 0 \\
\frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_i}\right) - \frac{\partial (T-V)}{\partial q_i} &= 0.
\end{aligned}$$

Then defining $T - V$ as the Lagrangian L, we have the desired result.

Example 12. Simple pendulum.



$$T = \frac{1}{2}ml^2\dot{\theta}^2, \quad V = -mgl\cos\theta$$

so

$$L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$$

Now

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= \frac{d}{dt}(ml^2\dot{\theta}) + mgl\sin\theta \\
&= ml^2\ddot{\theta} + mgl\sin\theta = 0.
\end{aligned}$$

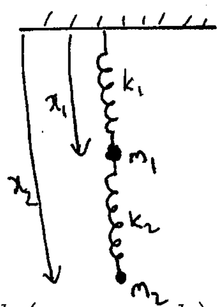
That is,

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0.$$

Example 13. Two particles on springs.

Let l_1, l_2 be the natural lengths of springs 1 and 2 respectively. Then

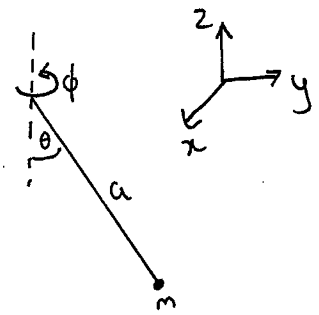
$$\begin{aligned}
L &= T - V \\
&= \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + m_1 g x_1 + m_2 g x_2 \\
&- \frac{1}{2}k_1(x_1 - l_1)^2 - \frac{1}{2}k_2(x_2 - x_1 - l_2)^2
\end{aligned}$$



Then

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} &= m_1\ddot{x}_1^2 - m_1 g + k_1(x_1 - l_1) - k_2(x_2 - x_1 - l_2) = 0 \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} &= m_2\ddot{x}_2^2 - m_2 g + k_2(x_2 - x_1 - l_2) = 0
\end{aligned}$$

Example 14. Spherical Pendulum.



$$\underset{\sim}{r} = a\cos\theta\,\underset{\sim}{\hat{k}} + a\sin\theta(\cos\phi\,\underset{\sim}{\hat{i}} + \sin\phi\,\underset{\sim}{\hat{j}})$$

$$\dot{\underset{\sim}{r}} = a\dot{\theta}\sin\theta\,\underset{\sim}{\hat{k}} + a(\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi)\underset{\sim}{\hat{i}} + a(\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi)\underset{\sim}{\hat{j}}$$

$$v^2 = \dot{\underset{\sim}{r}}\cdot\dot{\underset{\sim}{r}} = a^2\dot{\theta}^2 + a^2\sin^2\theta\,\dot{\phi}^2$$

$$V = mga(1-\cos\theta), \quad T = \frac{1}{2}m[(a\dot{\theta})^2 + (a\sin\theta\dot{\phi})^2]$$

So

$$L = \frac{1}{2}m[a^2\dot{\theta}^2 + a^2\sin^2\theta\dot{\phi}^2] - mga(1-\cos\theta).$$

In this case $L = L(\theta, \dot{\theta}, \dot{\phi})$ and is independent of ϕ. So Lagrange's equations give

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right),$$

Hence $\frac{\partial L}{\partial \dot{\phi}}$ is constant, so

$$\sin^2\theta\dot{\phi} = c, \quad c \text{ a constant.}$$

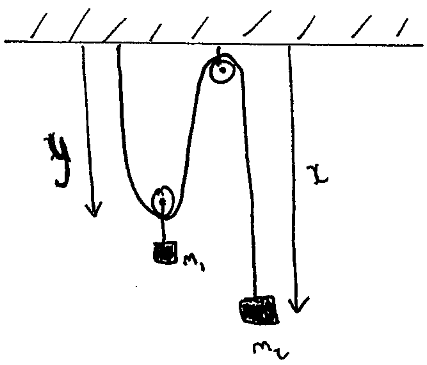
The other equation is

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = ma^2\ddot{\theta} - ma^2\sin\theta\cos\theta\dot{\phi}^2 + mga\sin\theta,$$

which we write as

$$\ddot{\theta} + \sin\theta(\cos\theta\dot{\phi}^2 - \frac{g}{a}) = 0.$$

Example 15. Pulley system.



Assume the rope and pulleys are frictionless and massless.

$$L = \frac{1}{2}m_1\dot{y}^2 + \frac{1}{2}m_2\dot{x}^2 + m_1gy + m_2gx$$

But $2y + x = c$ (the constant length of the rope). So

$$y = \frac{c-x}{2}, \quad \dot{y} = -\frac{1}{2}\dot{x}.$$

Then

$$L = \frac{1}{2}(\frac{m_1}{4} + m_2)\dot{x}^2 + g(m_2 - \frac{m_1}{2})x + \frac{m_1 gc}{2}.$$

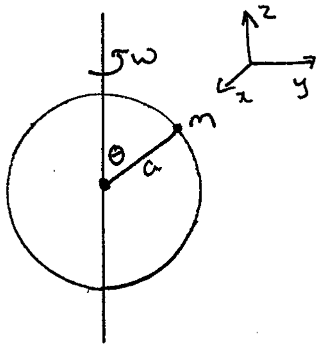
Thus our equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = (\frac{m_1}{4} + m_2)\ddot{x} + g(m_2 - \frac{m_1}{2}) = 0.$$

Rearranging, we find

$$\ddot{x} = \frac{(m_2 - \frac{m_1}{2})}{(\frac{m_1}{4} + m_2)}g.$$

Example 16. Bead on a rotating wire, where the wire is rotating with angular velocity ω.



$$V = mga\cos\theta, \quad T = \frac{1}{2}m[(a\dot{\theta})^2 + \omega^2 a^2 \sin^2\theta]$$

Therefore

$$L = T - V = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}m\omega^2 a^2 \sin^2\theta - mga\cos\theta.$$

Our equation of motion is

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = ma^2\ddot{\theta} - m\omega^2 a^2 \sin\theta\cos\theta - mga\sin\theta.$$

That is,

$$\ddot{\theta} - \sin\theta(\omega^2\cos\theta + \frac{g}{a}) = 0.$$