8 Stability

Often a system will have a steady-state solution where one or more variables remain constant. Assume a system has a steady-state solution $q_i = x$. Then we can learn about the behaviour of the system by solving the equations of motion for $q_i = x + \delta x$ where δx is a *small* perturbation.

In order to understand the nature of the solutions we usually make a *first-order Taylor approximation.* i.e.

$$f(x + \delta x) \approx f(x) + f'(x)\delta x.$$

This is valid provided δx is sufficiently small. Note that this is equivalent to stating

$$f'(x) \approx \frac{f(x + \delta x) - f(x)}{\delta x}$$

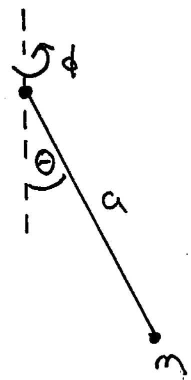
for small δx.

Example 21. Spherical Pendulum.
$V = mga(1 - \cos\theta)$, $T = \frac{1}{2}m[(a\dot{\theta})^2 + (a\sin\theta\dot{\phi})^2]$ and thus

$$L = T - V = \frac{1}{2}m(a^2\dot{\theta}^2 + a^2\sin^2\theta\dot{\phi}^2) - mga(1 - \cos\theta).$$

In this case $L = L(\theta, \dot{\theta}, \dot{\phi})$ is independent of ϕ which is thus a cyclic coordinate. The corresponding generalised momentum is

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = ma^2\sin^2\theta\dot{\phi} = \text{ constant.}$$



Also,

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ma^2\dot{\theta}$$

and satisfies

$$\frac{dp_\theta}{dt} = \frac{\partial L}{\partial \theta} = ma^2\sin\theta\cos\theta\dot{\phi}^2 - mga\sin\theta,$$

or

$$\ddot{\theta} = \sin\theta\cos\theta\dot{\phi}^2 - \frac{g}{a}\sin\theta$$

which is the equation of motion.

A solution to the equations of motion is given by circular motion at a fixed angle of inclination $\theta = \alpha$ (constant) provided that $\dot{\phi}$ is constant and

$$0 = ma^2\sin\alpha\cos\alpha\dot{\phi}^2 - mga\sin\alpha \quad \Rightarrow \dot{\phi}^2 = \frac{g}{a\cos\alpha}.$$

Thus the mass moves uniformly in a circle with angular velocity $\dot{\phi} = \omega = \sqrt{\frac{g}{a\cos\alpha}}$ where $-\pi/2 < \alpha < \pi/2$.

Stability: Consider a small perturbation away from circular motion so that $\theta = \alpha + \delta\theta$ with $\delta\theta$ small. Then the above equation for p_ϕ becomes

$$ma^2 \sin^2(\alpha + \delta\theta)\dot{\phi} = \text{const.} = ma^2\omega \sin^2\alpha,$$

or

$$\dot{\phi} = \frac{\omega \sin^2\alpha}{\sin^2(\alpha + \delta\theta)}.$$

Substituting this into the second equation of motion, we find

$$\ddot{\delta\theta} = \frac{\omega^2 \sin^4\alpha}{\sin^3(\alpha + \delta\theta)} \cos(\alpha + \delta\theta) - \frac{g}{a} \sin(\alpha + \delta\theta)$$

But for $\delta\theta$ small we have the first order approximations

$$\sin(\alpha + \delta\theta) \approx \sin\alpha + \cos\alpha\delta\theta, \quad \cos(\alpha + \delta\theta) \approx \cos\alpha - \sin\alpha\delta\theta$$

and

$$\frac{1}{\sin^3(\alpha + \delta\theta)} \approx \frac{\sin\alpha - 3\cos\alpha\delta\theta}{\sin^4\alpha}.$$

Hence

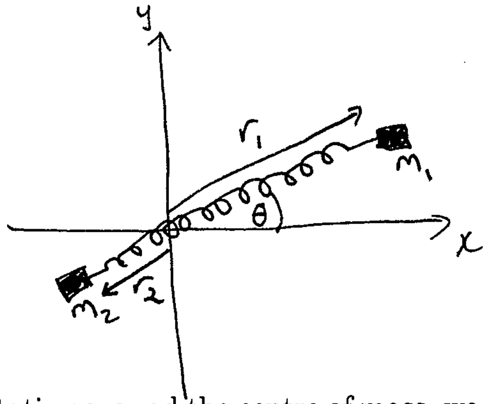
$$\begin{aligned}
\ddot{\delta\theta} &\approx \omega^2(\sin\alpha - 3\cos\alpha\delta\theta)(\cos\alpha - \sin\alpha\delta\theta) - \omega^2\cos\alpha(\sin\alpha + \cos\alpha\delta\theta) \\
&\approx -\omega^2(\sin^2\alpha + 4\cos^2\alpha)\delta\theta \\
&= -\omega^2(1 + 3\cos^2\alpha)\delta\theta.
\end{aligned}$$

Thus θ will oscillate around α with frequency $\omega\sqrt{1 + 3\cos^2\alpha}$ (simple harmonic motion), and the circular orbits are stable.

Example 22. Two masses m_1 and m_2 are connected by a (massless) spring of stiffness k. The system is set rotating about the centre of mass with an angular velocity ω and then released. If the masses are slightly disturbed along the line joining them, show that the angular frequency of oscillation is

$$\sqrt{\frac{3\omega^2 m_1 m_2 + k(m_1+m_2)}{m_1 m_2}}.$$

Solution: Let r_1, r_2 be the distances of masses m_1, m_2 respectively from the centre of mass.



As the system is rotating around the centre of mass, we know $m_1 r_1 = m_2 r_2$, i.e. $r_2 = \frac{m_1}{m_2} r_1$. Then

$$\begin{aligned} T &= \frac{1}{2}m_1(\dot{r}_1^2 + r_1^2\dot{\theta}^2) + \frac{1}{2}m_2(\dot{r}_2^2 + r_2^2\dot{\theta}^2) \\ &= \frac{1}{2}(1+\frac{m_1}{m_2})(\dot{r}_1^2 + r_1^2\dot{\theta}^2) \end{aligned}$$

and

$$V = \frac{1}{2}k(r_1+r_2-l)^2 = \frac{1}{2}\Big[r_1(1+\frac{m_1}{m_2}) - l\Big]^2,$$

so

$$L = T - V = \frac{1}{2}(1+\frac{m_1}{m_2})(\dot{r}_1^2 + r_1^2\dot{\theta}^2) - \frac{1}{2}\Big[r_1(1+\frac{m_1}{m_2}) - l\Big]^2.$$

L has no explicit dependence on θ, so θ is cyclic and $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m_1(1+\frac{m_1}{m_2})r_1^2\dot{\theta}$ is constant. i.e. $r_1^2\dot{\theta} = c$ for some constant c. Let the distance of m_1 from the centre of mass before the disturbance be d. Then $c = d^2\omega$, and $\dot{\theta} = \frac{d^2}{r_1^2}\omega$.

The other equation of motion is

$$\begin{aligned} 0 &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} \\ &= m_1(1+\frac{m_1}{m_2})\ddot{r} - m_1(1+\frac{m_1}{m_2})r_1\dot{\theta}^2 + k(1+\frac{m_1}{m_2})[r_1(1+\frac{m_1}{m_2}) - l] \end{aligned}$$

or

$$
\begin{aligned}
\ddot{r}_1 &= \dot{\theta}^2 r_1 - \frac{k}{m_1}[r_1(1+\frac{m_1}{m_2}) - l] \\
&= \frac{d^4}{r_1^3}\omega^2 - \frac{k}{m_1}[r_1(1+\frac{m_1}{m_2}) - l].
\end{aligned}
$$

As the masses are only slightly disturbed, we can set $r_1 = d + \delta r$ where δr is small. Then

$$
\frac{1}{r_1^3} = \frac{1}{(d+\delta r)^3} \approx \frac{1}{d^3} - \frac{3\delta r}{d^4} = \frac{d - 3\delta r}{d^4},
$$

using a first order Taylor expansion. Hence the second equation of motion becomes

$$
\begin{aligned}
\ddot{\delta r} &\approx (d - 3\delta r)\omega^2 - \frac{k}{m_1}[(d+\delta r)(1+\frac{m_1}{m_2}) - l] \\
&= -\left[3\omega^2 + \frac{k}{m_1}(1+\frac{m_1}{m_2})\right]\delta r + c' \qquad c' \text{ a constant} \\
&= -\left[\frac{3\omega^2 m_1 m_2 + k(m_1+m_2)}{m_1 m_2}\right]\delta r + c'.
\end{aligned}
$$

Hence the mass will oscillate with angular frequency

$$
\sqrt{\frac{3\omega^2 m_1 m_2 + k(m_1+m_2)}{m_1 m_2}}.
$$

Exercise: A particle moves on the inside surface of a cone of half angle α. The axis of the cone is vertical with the vertex downwards. Determine the condition on the angular velocity ω such that the particle can describe a horizontal circle h above the vertex. Show that the period of small oscillations about this circular path is

$$
\frac{2\pi}{\cos\alpha}\sqrt{\frac{h}{3g}}.
$$